SVENSKA GEOFYSISKA FÖRENINGEN

Tellus

A QUARTERLY JOURNAL OF GEOPHYSICS

VOLUME 2, NUMBER 1 FEBRUARY 1950

CONTENTS

Page

H. RIEHL — On the Role of the Tropics in the General Circulation of the Atmosphere 1

S. PETTERSSEN — On the Sensitivity of the Wind Field Relative to Pressure Variations 18

H. NORINDER — Total Duration of Lightning Strokes and Occurrence of Multiple Strokes 24

P. QUENEY — Adiabatic Perturbation Equations for a Zonal Atmospheric Current..... 35

J. VAN MIEGHEM — Sur la circulation transversale associée à un courant atmosphérique 52

A. M. CHAUDHURY — On the Vertical Distribution of Wind and Temperature over Indo-Pakistan along the Meridian 76° E in Winter 56

Shorter Contribution

F. LINDHOLM — Atmospheric Effects on Cosmic Radiation, Air-mass Absorption and Meson Decay..... 63

Book Review

B. GUTENBERG and C. F. RICHTER: Seismicity of the Earth and Associated Phenomena, reviewed by M. BÅTH 68

Printed and Distributed by Generalstabens Litografiska Anstalt

Stockholm, Sweden 1950

Tellus

Editor

C.-G. ROSSBY

Stockholms Högskolas Institut för Meteorologi, Stockholm

Assistant editor

C. C. WALLÉN

Sveriges Meteorologiska och Hydrologiska Institut, Stockholm

Advisory Board

H. W:SON AHLMANN, Stockholm. H. ALFVÉN, Stockholm. N. AMBOLT, Stockholm. T. BERGERON, Uppsala. V. W. EKMAN, Stockaryd. P. GEIJER, Stockholm. H. HEDSTRÖM, Stockholm. G. ISING, Djursholm. H. KÖHLER, Uppsala. H. NORINDER, Uppsala. H. PETTERSSON, Göteborg. O. RYDBECK, Göteborg. G. SLETTENMARK, Stockholm. N. ZEILON, Lund. A. ÅNGSTRÖM, Stockholm.

Tellus is a medium for the publication of original contributions, survey articles and discussions in the field of the geophysical sciences. It is published quarterly by the Swedish Geophysical Society, under a cooperative agreement between the Society and Generalstabens Litografiska Anstalt and with the aid of a grant from the Swedish Government. Through the publication of Tellus the Swedish Geophysical Society wishes to contribute to international cooperation between geophysicists and to provide an international forum for discussions and exchange of information. To serve these purposes, all contributions will be published in English, French or German.

Contributions should be sent directly to the editor at Sveriges Meteorologiska och Hydrologiska Institut, Fridhemsgatan 9, Stockholm. Authors may be of any nationality but only manuscripts in English, French or German can be accepted.

The categories in Tellus are articles, shorter contributions and announcements, correspondence and bookreviews. A shorter contribution usually does not exceed two printed pages. Correspondence is intended for discussion of published papers. Ordinarily each letter should not exceed more than about half a page.

The manuscript must be complete and final when submitted. It should be an original typewritten copy, double or triple spaced with wide margins and placed on one side only and all pages should be consecutively numbered. Only one copy of the manuscript need be submitted.

An abstract is required at the beginning of each article but not with a shorter contribution.

Each paper must be fully documented as to references. These should be grouped at the end of the paper and arranged alphabetically by authors' names. Each reference should be complete and in the following order. For an article: author(s), year, title of article, title of serial publication (underlined), volume number, pages. For a book: author(s), year, title of book (underlined), city of publication, publisher, pages.

The illustrations should accompany the manuscript and be in final form. Each figure should be mentioned specifically in the text and should have an adequate legend. Figure number and legend should not be part of the drawing, but, for identification, the figure number should be written outside the border of each drawing. A typed list of the legends should be furnished separately. Originals are preferably made about twice final size. Lettering must be large enough to remain clearly legible when reduced. Drawings submitted should be originals, drawn entirely in black India ink. Maps are preferably so oriented that the meridian through the center of the drawing is parallel to the sides.

Mathematical work should be carefully written with pen or typewriter, leaving a wide space above and below each character for marking for the printer.

Authors of articles receive 50 reprints free of charge. Additional reprints must be paid for by the author. Forms for the request of reprints will be enclosed with the galley-proofs. No reprints are provided of shorter contributions, correspondence and bookreviews.

The subscription price is U. S. $ 6.00 per year (U. S. $ 2.00 per single issue) or equivalent amount in Swedish currency according to the prevailing official rate of exchange. Remittances should be made payable to Dr C. J. Östman, Tellus, and sent to the office of the editor. Members of the Swedish Geophysical Society receive one copy each of Tellus at a reduced rate; the necessary arrangements should be made with the Treasurer of the Society. Subscription orders, claims for missing numbers, changes in address and other business correspondence should be addressed to the office of the editor, Fridhemsgatan 9, Stockholm, Sweden.

Subscription orders from booksellers should preferably be sent directly to Generalstabens Litografiska Anstalt, Förlaget, Stockholm 1, Sweden.

Ansvarig utgivare: A. Ångström

SVENSKA GEOFYSISKA FÖRENINGEN

VOLUME 2, NUMBER 1 **Tellus** FEBRUARY 1950

A QUARTERLY JOURNAL OF GEOPHYSICS

On the Role of the Tropics in the General Circulation of the Atmosphere

By HERBERT RIEHL, University of Chicago

(Manuscript received 26 September, 1949)

Abstract

Contrary to older descriptions, the air currents in the middle and upper troposphere over the low latitudes are not steady but highly variable. There are trains of well-developed waves or vortices. When these disturbances combine with troughs in the polar westerlies, the resulting extended troughs often reach from pole to equator. Poleward flow of heat must take place mainly along these troughs. Energy from low latitudes is injected into the temperate zone in narrow and variable areas only.

A description of the tropical atmosphere in the layers below and above the bases of the cumuli leads to an attempt to explain the steadiness of the low-level wind field which contrasts with the high variability of the upper layers. Then the strength of the mean meridional circulation and the fields of divergence and convergence at the surface are calculated, followed by a summary of available information on the high-level circulation. The field of motion aloft varies greatly with longitude, so that inferences concerning the tropics as a whole can only be made from maps covering most longitudes. Such maps (300-mb charts) are available for a limited period down to latitude 10° N. They indicate that regions where the easterlies decrease upward alternate with regions of increasing easterlies.

Introduction

For many years, the belief has held sway that the presence of atmospheric disturbances is peculiar to middle latitudes, whereas steady conditions prevail in the arctic and in the tropics. This conviction helped to reduce the excessive number of variables which confront the meteorologist at every turn. In the tropics, it found strong support in the apparent monotony of surface climatic conditions over wide areas, reflected mainly in small variability of temperature and wind. The opinion prevailed that conditions in the free atmosphere were as steady as those near the ground. Therefore, with the aid of few and scattered upper-air data it was held possible to deduce the correct circulation picture aloft. On any given day the upper flow should be indicative of average conditions, in contrast to middle latitudes. Solar energy accumulated in the tropics ultimately must sustain the rapid westerly flow of higher latitudes. Yet, because of the steadiness of the tropical regime, short-period fluctuations of kinetic energy outside the tropics are not traceable to variations in the

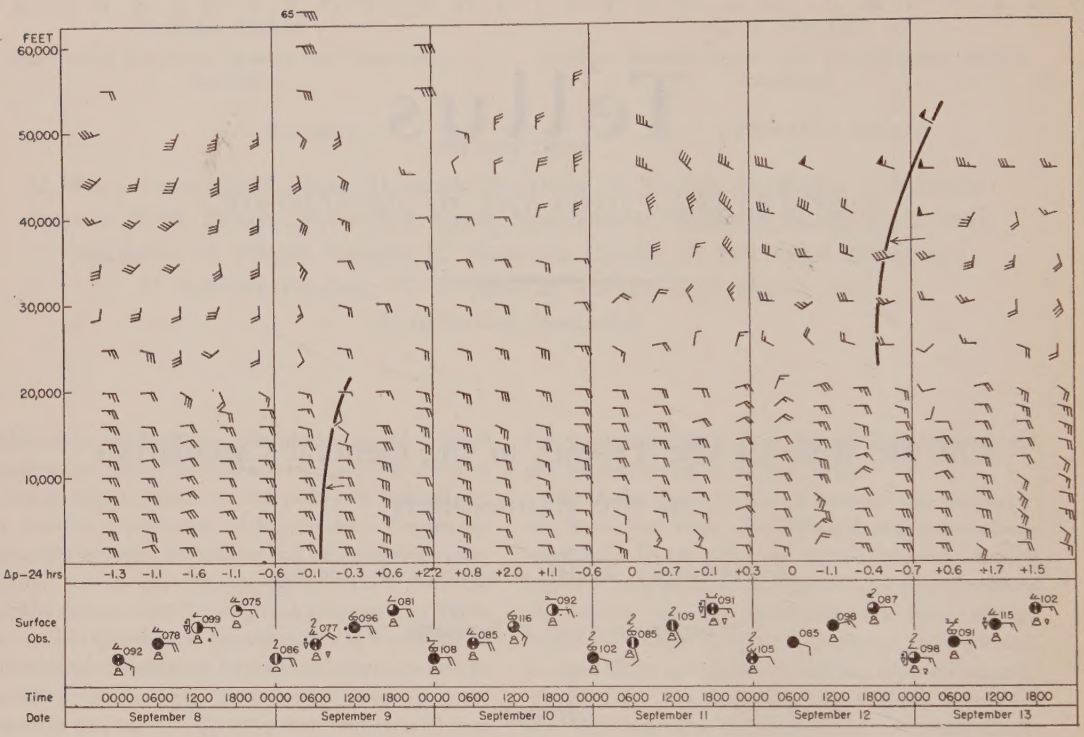


Fig. 1. Vertical time section at Eniwetok (11° N, 162° E), September 8—13, 1945 (Greenwich time). A long barb represents 10 mph, a short barb 5 mph and a heavy triangular barb 50 mph. Heavy solid lines are troughlines. 24-hour surface pressure changes are plotted above the surface observations.

source region of the energy. Poleward transport of heat energy across the subtropical ridgeline is subject only to seasonal and long-period variations.

Gradual accumulation of aerological data, however, has revealed that space and time variations, especially of the wind flow aloft, are surprisingly large. Even the pre-war literature carries the annoyed comment that the more numerous the observations from the tropics, the more complicated tropical analysis. The large amount of high-level data collected in most parts of the tropics during the recent war finally has made it amply clear that in many low latitude regions the variability of winds aloft equals that of the temperate zone.

This is true notably in the upper troposphere. Let us glance at a time cross-section of rawin observations taken at Eniwetok in the Marshall Island group of the Pacific Ocean (fig. 1), a section typical of the sequence of events aloft over wide areas of the tropics. In the lower atmosphere, up to about 600-mb, the steadiness of the flow is undeniable. From there on upward fluctuations increase. *Extreme restlessness rules in the upper troposphere*, to subside only as we pass across the tropopause into the stratosphere. The classical notion about the tropics thus is partly right — but it gives only half of the story. In the low troposphere the course of events is very uniform, especially the important intake of heat and moisture from the oceans which is dependent in part on the surface wind. This, however, cannot hold for the poleward discharge of heat aloft. At the very least, this heat flow must vary from longitude to longitude as north and south winds alternate along the latitude circles.

A hemispheric map of the upper troposphere (fig. 2) brings this out even more clearly. For several months toward the end of the last war simultaneous upper-air observations were taken throughout the tropical and equatorial belts of the northern hemisphere: over Africa, the Americas, the Pacific Islands, and south-

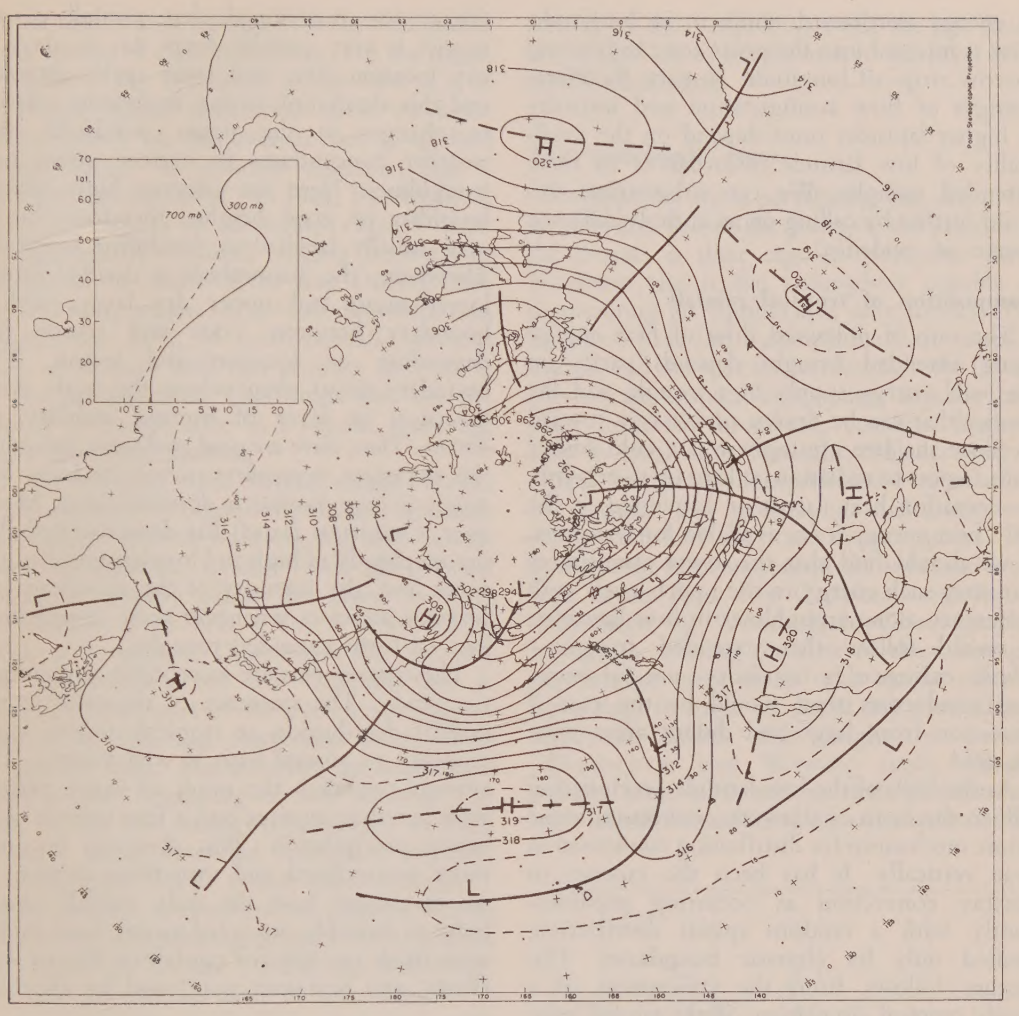


Fig. 2. Topography of the 300-mb surface (100's feet), August 26, 1945, 0600Z. Insert: meridional profile of the zonal wind (mps) at 300 and 700 mb computed between longitudes 20° E and 110° E via western hemisphere.

eastern Asia. These months are eminently suited for large-scale analysis. Under auspices of the Office of Naval Research, United States Navy, the entire period was analyzed at the University of Chicago, both at low and high levels. The chart for August 26, 1945, represents a random selection from this file. Even under the favorable circumstances prevailing in 1945 station coverage was poor in some areas, especially in the tropical Atlantic. Differential analysis therefore was a standard auxiliary analysis tool, together with time cross-sections and other techniques employed under such conditions. It is not possible to vouchsafe for all details of fig. 2. But its broadscale features are submitted without hesitation as reliable.

The most outstanding of these features are the *breakdown of the tropical atmosphere into a train of vortices* (RIEHL 1948) *and the complete interlocking of flow between high and low latitudes.* It would seem more than difficult to draw a line separating polar and tropical zones. Everywhere, troughs of great extent — called »extended troughs» by CRESSMAN (1948) — reach from pole to equator. It is only to the east of these troughlines that heat can flow poleward. Thus there is not a general seepage

of energy northward, uniform in longitude. Heat is injected into the polar zone in few and narrow strips of longitude. In part, therefore, changes of flow configuration and intensity in higher latitudes must depend on the availability of low latitude disturbances to form extended troughs. We can substantiate this point further by calling on an entirely different source of evidence.

Composition of tropical rainfall

The rate of poleward flow of heat energy along extended troughs depends partly on the local energy supply in a trough, and the energy that can be drawn advectively toward it. Since the free atmosphere is a cold source with respect to radiation, at least up to 200-mb, condensation heat, latent or converted, is the only heat energy form available for transport. If we should find that the lower atmosphere transmits such energy to the upper levels with preference along troughlines in low latitudes, it would follow that extended troughs—whose existence is transitory—are far better heat conductors than would be the case if advection from large and distant areas were required.

As the bulk of the low latitude precipitation falls in the form of showers, convection is the prime mechanism for distributing condensation heat vertically. It has been the custom to portray convection as occurring predominantly with a random spatial distribution, limited only by climatic boundaries. This picture follows from the conception of a steady tropical circulation. Slight rainfall variability should accompany the minute fluctuations of surface temperature and wind. As is well known, this does not hold true, even from the seasonal viewpoint. In any given area, excess rainfall may flood the fields in one year and drought may strike in the same season of the next year. If we determine in what manner the individual days contribute to the rainfall total, the discrepancy with the classical description becomes even more striking, even though we hear that in many locations showers begin every day at a certain hour with clocklike regularity, and even though the Hawaiian tongue contains no word for "weather".

Along with many others who have ventured to dispute the classical viewpoint, this writer (1945) has maintained that rainfall of the tropics is very variable from day to day. At any location clear and rainy spells alternate, and this alternation occurs in definite relation to changes in the upper windfield. Bad weather concentrates in narrow zones near troughlines. Here we observe high relative humidity to great heights, therefore the air mass usually labelled as "maritime tropical." Elsewhere, the atmosphere is divided into a lower moist and upper dry layer, with a boundary between 5,000 and 10,000 feet depending on location and season. This boundary exists even where the trade wind inversion or layer of thermal stability are absent. *Thus, there are good qualitative indications that the upper troposphere receives its latent heat supply in large measure in disturbed zones.* Moreover, CRESSMAN (1948) has demonstrated that the amplitude of high and low latitude disturbances and the intensity of the associated bad weather increases whenever these disturbances join to form extended troughs.

These statements are susceptible to quantitative study. The existence of skewness in the rainfall distribution at tropical stations on a monthly or annual basis is well known. The average rainfall is the result of many periods with less than average and a few periods with heavy precipitation. This skewness becomes more pronounced and sometimes extreme if we investigate how the daily rainfall contributes to monthly or annual means. Such studies were made recently for rainfall on Puerto Rico (RIEHL and SCHACHT 1947) and for Florida[1].

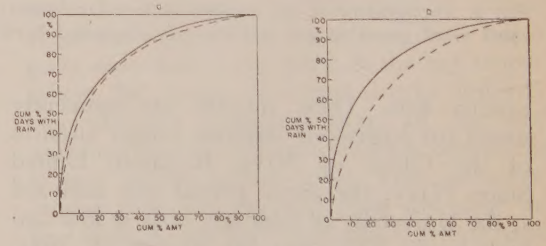


Fig. 3. Cumulative percentage frequency distribution of rainfall at San Juan, Puerto Rico (solid curves) and for average of all stations on Puerto Rico (dashed curves): a) for dry season, February—March; b) for rainy season, August—September. Ordinate: percent of days with rain; abscissa: percent of rainfall. Ten years were used to compute diagram. Hurricane precipitation is not involved.

[1] H. RIEHL: "Florida thunderstorms and rainfall." J. Meteor. **6**, 289—290, letter to editor.

Similar results were obtained for the Philippines in the course of class instruction. On the island of Oahu in the Hawaiian group over 2/3 of the annual rainfall stems from about 10 major storms which on the average last 3—4 days (RIEHL 1949). As another example, fig. 3 illustrates conditions at San Juan, Puerto Rico. This city enjoys a moderate climate with average monthly rainfall ranging from 2.5 to 7 inches in the course of the year. During both dry and rainy season 10 % of all days with rain account for 50 % of total rainfall, whereas 50 % of all days with rain produce only 10 % of the precipitation. This holds also for the rainfall of Puerto Rico as a whole, including many mountain stations.

It follows that most tropical rainfall occurs in consequence of organized, not unorganized, convection. *Within relatively few and narrow zones of disturbed weather does the tropical atmosphere actually obtain the largest share of that portion of its heat which is derived from release of latent heat.* These zones, the "secondary" disturbances of low latitudes act as tropical counterpart of the cyclones of higher latitudes in effecting conversion from latent to sensible heat. This confirms the hypothesis raised at the beginning of this section. The bulk of the heat energy transported to high latitudes along extended troughs is injected into the upper levels within these troughs.

Our attempt to demonstrate that the tropics are a factor in producing changes of the general circulation so far has yielded the following: the field of motion aloft in low latitudes is marked by great restlessness connected with the passage of vortex or wave trains; heat flows poleward in a few narrow channels, variable in longitude and time; the heat available for transport is deposited in the upper air in a few small areas. These observations also imply that it is not possible to arrive at an understanding of the general circulation by employing only steady state dynamics applied to a statistical mean circulation. This was recognized early by DEFANT (1921) for the higher latitudes. ROSSBY's studies (cf. ROSSBY 1949) place increasing emphasis on the individual members that compose the mean circulation. Outside the tropics, the flow of air is variable in space and time at all heights. It is a peculiarity of the tropics that for the most part only the high atmosphere is characterized by this variability, while the low levels are steady. Let us briefly turn to this layer with steady circulation.

The "subcloud" layer

Perhaps the most perplexing fact about the air currents below the bases of the cumuli in the tropics is their regular monotone flow. Nowhere else is the atmosphere capable of maintaining an equilibrium state locally. The intake of heat and moisture from the tropical oceans, and the heat accumulation in the lowest part of the tropical atmosphere actually take place in steady broadscale currents, featured only by small-scale vertical turbulence. JACOBS (1942) has made comprehensive calculations of seasonal and geographic distribution of energy transfer from ocean to atmosphere. The most recent information on the subcloud layer itself comes from two Caribbean expeditions undertaken by Woods Hole Oceanographic Institution. During the first of these voyages experiments were made with smoke released from ships in order to detect any vertical and horizontal eddies that might be present (WOODCOCK and WYMAN 1946). Smoke plumes were observed to drift downstream with lateral distortion of the orientation of the

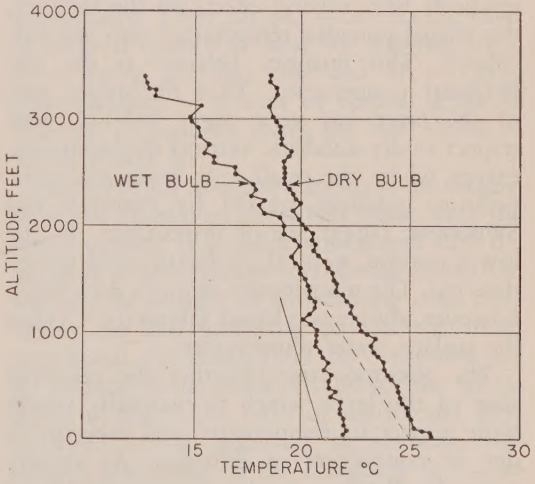


Fig. 4: Vertical structure of the subcloud layer near Puerto Rico on April 2, 1946. Airplane ascent was made about noon local time. Light dashed line is dry adiabatic line, light solid line is line of constant specific humidity. Reproduced with permission of Woods Hole Oceanographic Institution.

plumes from the mean wind direction that attained over 45°, whereas the calculated crosswind varied by 5° at most. In the vertical, smoke rose up to 150 m and successive upward bulges were about 300 m apart. This is taken as evidence that the plumes are floating in a convection cell pattern that with simplification can be patterned on the Bénard cell.

The second expedition, led by J. Wyman and A. H. Woodcock, took ship and aircraft measurements of temperature, humidity, turbulence, and vertical motion from the vicinity of the sea surface up to the trade wind inversion near Puerto Rico during April 1946. A study of the trajectories of the air sampled showed it had reached the Antilles on paths mainly from the north-central Atlantic (Bunker, Haurwitz, Malkus, Stommel 1949). Certain differences compared to the data of the "Meteor" expedition therefore should be expected. In particular, the large superadiabatic lapse rates of the eastern Atlantic should have disappeared. This was indeed observed. The subcloud layer near Puerto Rico turned out to be almost completely mixed (fig. 4) and can be regarded as such for all large-scale atmospheric problems. In the upper half of the layer the potential temperature increases very slowly with height (about 0.2°C/100 m). Conversely the mixing ratio decreases. As these gradients have inverse effects on the stability, the virtual potential temperature also was calculated. This quantity behaves as the dry potential temperature. Thus the upper part of the layer has very slight stability with respect to dry-adiabatic vertical displacements, except where the small-scale horizontal temperature gradients treated by Priestly and Swinbank (1947) are of importance. In the lower portion, vertical gradients could not be detected. The temperature at ship's deck level, however, always was lower (about 0.5°C) than the surface water temperature.

The question arises whether the observed state of the layer which is essentially steady with respect to temperature and moisture is due to convection or diffusion. As already seen, the Panama smoke experiments favor convection as does the temperature difference between ocean and deck level. Yet the upward increase of virtual potential temperature in the upper part of the layer speaks against this solution. Moreover, turbulence records taken

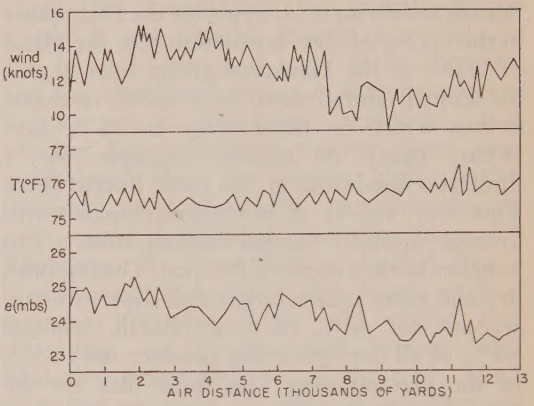


Fig. 5. Fluctuations of surface wind, temperature and vapor pressure on April 23, 1946, as obtained on ship stationed near Puerto Rico. Abscissa gives time scale converted to horizontal distance to indicate eddy sizes. The variations of vapor pressure are caused by fluctuations of water vapor density as the temperature effect is very small. Reproduced with permission of Woods Hole Oceanographic Institution.

in horizontal traverses showed only small turbulence gradients, and not even underneath the clouds themselves was there an increase of turbulence. These latter observations support the alternative, that diffusion is the governing mechanism. This is also suggested by the fact that throughout most of the subcloud layer the small-scale fluctuations of temperature and humidity mainly are out-of-phase (Langwell 1948) whereas at deck level they are in phase (fig. 5). In a convective atmosphere positive correlation should prevail generally.

Fig. 5, however, also indicates presence of larger eddies. Superposition of a large eddy on more minute fluctuations is easily discernible. The possibility, if not the likelihood remains that vertical motion gradients exist even in absence of turbulence gradients. If we postulate the presence of patterns of horizontal convergence and divergence that have the dimension of convection cells, it is still possible to seek the "roots" of the cumuli in the subcloud layer as a whole.

The Woods Hole observations point to a conceivable explanation for the steadiness of the windfield in the subcloud layer. Rossby has emphasized[1] that the development of large lateral turbulence eddies depends on the ther-

[1] Verbal discussions with the writer. The arguments based on the relation between vertical and horizontal eddies are included with permission of Professor Rossby.

mal stratification of the atmosphere. As the field of motion always seeks to attain gradient wind equilibrium, pressure gradients that can sustain the windfield must be able to develop aloft. On account of hydrostatic considerations this is possible only in the presence of horizontal temperature gradients that can be intensified or decreased. When lapse rates tend toward the adiabatic, vertical circulations predominate and preclude the formation of horizontal temperature gradients in the air over the oceans that are in excess of the temperature gradient of the sea surface. The lateral eddies then must recede in importance. This is precisely the situation that exists in the trade wind belt. A hypothesis can be formulated stating that because of lack of thermal stability over millions of square miles, extreme smallness of oceanic temperature gradients, and effectiveness of vertical turbulence, horizontal circulations will be held to a minimum. Flow in the low levels should be quasi-uniform, especially under trade wind inversions, since they shield the low troposphere from any disturbing influence of the high levels.

The cloud layer

The role of the tropical cumuli in transporting heat upward has held a place of prominence in papers on the general circulation for many years. It forms a cornerstone of most theories on that subject. Most reasoning is patterned to conform to the principles of the "parcel method" as prime mechanism of convection. The air that has travelled equatorward from the polar zones near the surface is thought to rise in random convective motion in the tropics. These buoyant parcels carry heat even to the high troposphere to compensate for radiation losses.

In recent years, dissatisfaction with the parcel method has become widespread. In particular, the observed temperatures inside cumuli are far colder than predicted by this approach. Modification of the theory of convection, begun with introduction of the "slice method" by J. BJERKNES (1938), has been carried forward recently in a brilliant paper by STOMMEL (1947). He explains the observed temperature and humidity lapse rates in cumuli with the assumption that a buoyant jet entrains air from all sides during its ascent.

Apart from the dynamics of convection, it is relevant to ask whether random convective heat transfer, as utilized in the older general circulation theories, is an effective mechanism under any circumstances. Consider a mass of air M_1, contained between the top of the subcloud layer and, say, the 300-mb level. This air is to be maintained at an equilibrium mean temperature T. If M_2, with mean temperature T_2, is the mass of air added from below in the course of one day to offset radiational heat losses, the total mass (M) aloft after 24 hours must be the sum of M_1 and M_2. Mass continuity is provided following classical theory by low-level inflow from higher latitudes and poleward outflow aloft. From the law of conservation of energy it follows that

$$MT = M_1 (T - \Delta T) + M_2 T_2,$$

where ΔT is the heat loss in 24 hours. The ratio

$$M_2/M_1 = \Delta T/(T_2 - T),$$

where $T_2 - T$ measures the latent heat of condensation realized in the course of one day. Let $T_2 - T = 5°\text{C}$, a large value seldom realized even in parcel method computations over the pressure interval considered, and let $\Delta T = 1°\text{C}$, a moderate amount to judge by what is known on radiational cooling in the troposphere. Then $M_2/M_1 = 0.2$. As the bases of tropical cumuli lie near the 950-mb level, $M_1 = 650$ g per unit area and $M_2 = 130$ g, corresponding to a layer of vertical depth of 130 mb, or roughly twice the total depth of the subcloud layer.

Since vertical mass movements of the magnitude just computed must take place over the entire tropics, it is easily anticipated that the strength of the average meridional circulation will not be capable of maintaining mass continuity. In the following section the mean meridional circulation will be computed for the surface. If the result is valid for the whole subcloud layer, the ratio of meridional flow observed to that necessary to maintain heat balance according to the foregoing is $^1/_{10}$ to $^1/_{20}$. *It follows that the average equatorward flow cannot be a factor of any account in providing a mechanism for heat transport from lower to upper troposphere.* We must fall back on turbulent eddy motion.

There are two principal choices for the dimension of these eddies. They may have the size of convection cells where the vertical thermal stratification gives rise to the motion. Or they may be "secondary" disturbances where the field of vertical motion is controlled by the horizontal. The answer has already been given. Yet, as the tropical troposphere in its entirety is unstable with respect to moist adiabatic ascent — apart from trade wind inversions — we might have expected a random distribution of cumulonimbi all over the tropics, surrounded by clear areas of descending air. If this were observed generally, uniform motion might prevail at high levels as it does near the ground. Actually the closely spaced trade cumuli have their tops between 6,000 and 10,000 feet. Above this level clouds occur rather strictly in "organized convection zones", although with "Feinstruktur". *It is a remarkable coincidence that the steadiness of the trade begins to decrease upward rapidly just above the tops of the trade cumuli.*

STOMMEL (1947) to date has offered the only acceptable explanation for the constantly observed restraint upon the growth of cumuli in spite of existence of positive areas on thermodynamic charts that often are huge. AUSTIN (1948) also has pointed out that because of entrainment large density gradients between cumuli and their surroundings should not occur. An explanation for the calculated entrainment rates has not yet been offered. Yet they are evidently of sufficient magnitude to prevent growth of random convective cells to great heights all over the tropics. The average height of the tops of trade cumuli outside of areas with strong trade wind inversion must be indicative of the rate of conversion of heat and kinetic into potential energy by entrainment. Because of this conversion upward heat transfer in vertical convection cells does not carry to the upper troposphere. This compels the development of an entirely different class of disturbances —large-scale horizontal eddies in the form of vortex trains—which funnel heat aloft in zones of organized convection. The vertical stability necessary to permit lateral concentration of isotherms aloft to support the windfield is produced by the small-scale lateral turbulence in the low levels that finds its expression in the form of entrainment.

The average circulation at the surface

We shall now carry out the calculation of the strength of the average meridional circulation at the surface. For this purpose, the extensive sum-

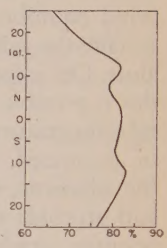


Fig. 6. Percent of circumference of globe covered by ocean between latitudes 25° N and 25° S.

maries of resultant winds con tained in the Atlas of Climatological Charts of the oceans are entirely suitable. The region selected extends from 25° N to 25° S. About 80 % of this area is covered by water (fig. 6). Since the mean circulation over the continents, especially Africa and Australia, resembles that over the oceans (cf. BRUNT 1939, pp. 14—15), it is reasonable to conclude that the following diagrams are valid for the whole equatorial belt.

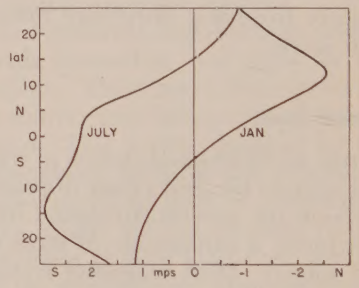


Fig. 7. Average meridional wind speed (mps) at the surface between latitudes 25° N and 25° S for the area given by Fig. 6.

The strength of the meridional flow (fig. 7) which indicates the net non-geostrophic mass transport across the latitude circles, is 1—2 mps or 1—2° latitude per day. Air crossing the 25th parallels N and S will require 2—3 weeks to reach the equator. It is relevant to ask whether the curves of fig. 7 represent small residual differences between alternating regions of northward and southward motion around the circumference of the globe on any latitude circle. Figs. 8 a and 8 b show that for the most part this is not the case, especially in July. In that month, winds with south component cover the entire oceanic area almost to 10° N, and the speed of the south component exceeds that of the north compo-

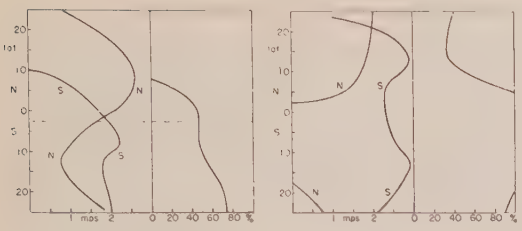


Fig. 8 a. North and south components of surface wind (mps) averaged on each latitude circle for all areas with north and south component separately: January. Right hand portion of diagram indicates percentage of circumference on each latitude circle that has south wind.

Fig. 8 b. Same as Fig. 8 a, for July.

nent. Flow of mass into the northern hemisphere amounts to about 1 % of the total mass of a hemisphere in 5 days, if the inflow extends through a layer 500 m deep.

The boundary between net transport from north and south lies near 2° S in January and shifts to 15° N in July. This displacement furnishes perhaps the fairest measure of the seasonal shift of the equatorial axis of the general circulation. It amounts to about 20° latitude whereas the sun oscillates over twice this latitude range. This fact has been taken by Rossby (1949) to suggest that the low latitude circulation pattern cannot be altogether thermally determined. Dynamical modifications must restrict the seasonal circulation changes to a smaller latitude range than the solar movement.

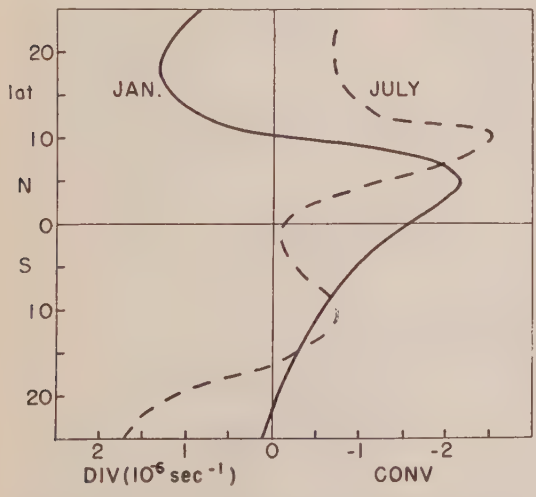


Fig. 9. Average velocity convergence and divergence from 25° N to 25° S for January (solid line) and July (dashed line). This diagram was computed by differentiation of Fig. 7 with respect to latitude.

Fig. 7 bears out the contention, made earlier, that the average meridional flow is an insignificant factor for the vertical heat transfer. Nevertheless, the computed meridional velocity gradients are quite appreciable. This can be seen if the curves of fig. 7 are differentiated with respect to latitude to obtain the field of divergence and vertical motion (fig. 9). The magnitude of the divergence is 10^{-6} sec^{-1}. Vertical displacements of 300—500 m per day would result at 3 km if the divergence was uniform between the ground and this level. These values are excessive. It is reasonable to suppose that the meridional wind component weakens rapidly in the friction layer, a contention supported by windroses at many stations. If the divergence decreases linearly with height and becomes zero at 900 mb, the vertical motion through the 900-mb surface has the order of 100 m/day, a reasonable value.

The principal region of convergence and upward motion lies in the northern hemisphere in both seasons. It experiences only a slight latitudinal shift and a surprisingly small change of intensity in the course of the year. A pronounced secondary convergence zone is evident in July but missing in January. It is interesting to note that surface divergence of considerable magnitude occurs in both subtropical belts during winter. Descent must be taking place in the cloud and subcloud layers underneath the trade wind inversion.

This is confirmed by a glance at the regional distribution of surface convergence and divergence (figs. 10, 11). As should be expected, a close correlation exists between the configuration of these charts and rainfall distribution over the oceans. In addition to the broad areas of divergence in the subtropics, the elongated narrow band of maximum convergence stands out that marks the equatorial convergence zone. Its intensity is almost uniform in both seasons as also noted from fig. 9. In the Indian Ocean, a pronounced secondary convergence zone extends near latitude 10° S in July. There is no trace of a split in January. In the Pacific, however, two convergence zones are evident in both seasons. Between them, the famous dry area of the equatorial

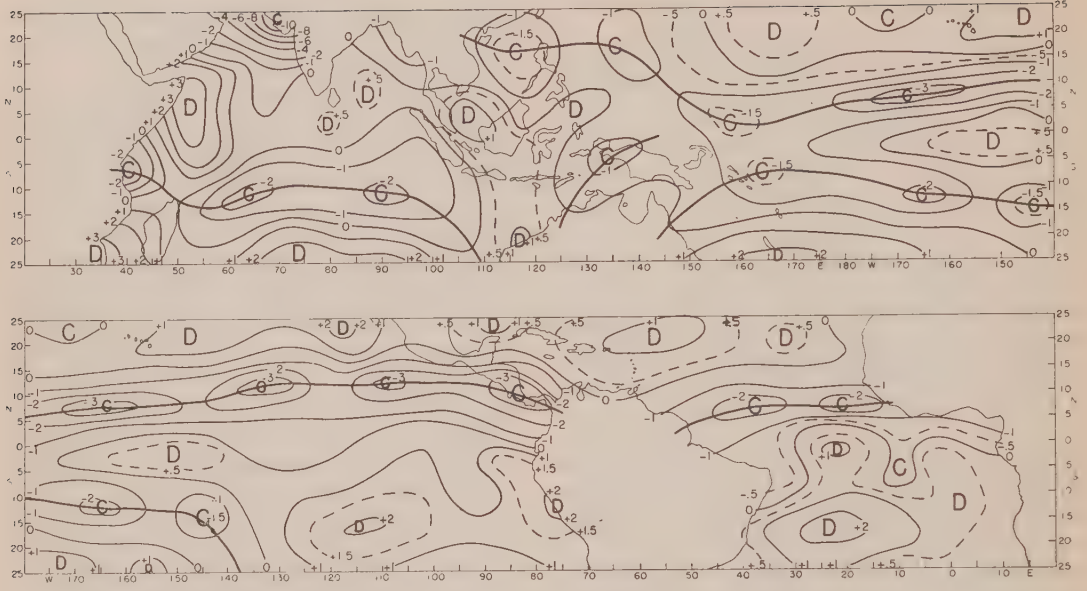


Fig. 10. Regional distribution of convergence and divergence (10^{-6}sec^{-1}) in January.

Pacific is very well marked. Thus the double convergence zone is a local feature of central and western Pacific during January. It extends over half of the equatorial belt in July, while the other half does not experience it as an average condition at any time.

The circulation in the upper troposphere

Interest has centered for many years on the vertical wind distribution over the tropics. Classical theory demands that at a short distance from the equator the easterlies begin to

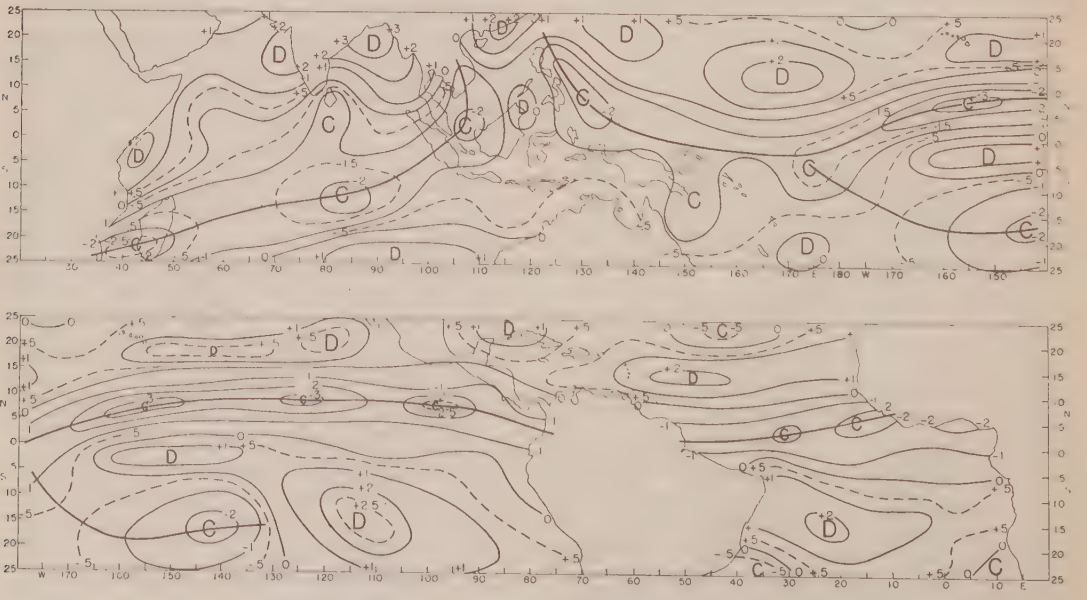


Fig. 11. Same as Figure 10, for July.

decrease with height and that a net poleward component of motion exists. In view of the preceding section, the latter requirement is undoubtedly fulfilled. It is not an object of this report to enter into the numerous theoretical arguments that are being advanced to explain the observed distribution of zonal momentum aloft. Here, we shall inquire merely concerning the state of knowledge on the actual windfield of the upper troposphere. On this point, contradictory statements have appeared of late.

Upper-wind summaries published in the course of the years have shown that in some locations the lower easterlies give way to upper westerlies during all seasons (cf. FASSIG 1933, STONE 1942). In other regions the easterlies attain their greatest strength in the upper troposphere at least during part of the year (India, Dutch East Indies, Philippines). When KUHLBRODT (1928) published his famous diagram of the vertical wind distribution over the Atlantic he stated that westerly winds replace the easterlies aloft. In the eastern Pacific Ocean, wind data at Hawaii show predominance of upper westerlies throughout the year.

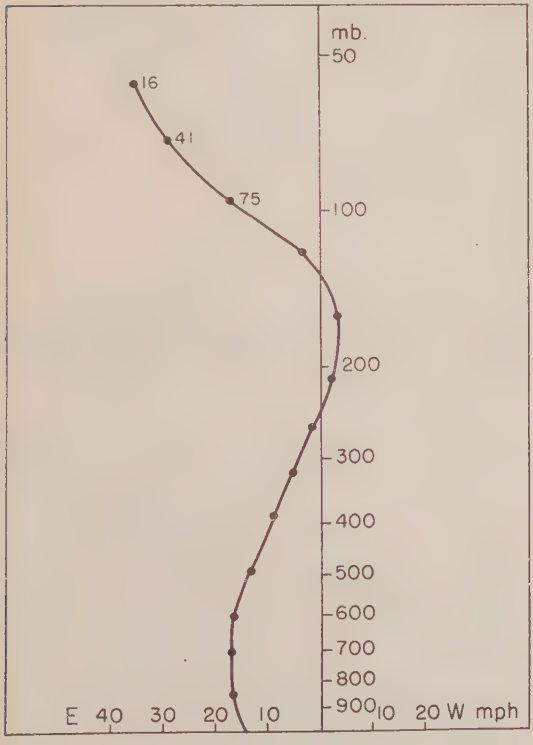


Fig. 12. Vertical variation of the basic zonal flow in the Pacific Ocean during September 1945, as determined from rawin observations. Diagram from RIEHL (1948).

A recent summary of rawin observations over the tropical Pacific during September 1945 (RIEHL 1948) also indicated upward decrease of the easterlies (fig. 12). HUBERT (1949) found strong westerlies at 200 mb in individual situations over the equatorial Pacific (Tarawa 1° N, 174° E; Phoenix Islands 4° S, 172° W). On the other hand, computations of a mean cross-section along the meridian 80° W led HESS (1948) to state: "In low latitudes the tropospheric isotherms are practically horizontal in both winter and summer. This lack of sleonoids in the vertical plane would seem to render untenable an explanation of the maintenance of the trade winds by north-south solenoids." The section of Hess shows easterlies increasing with height in the northern hemisphere summer, especially near latitude 20° N. VUORELA (1948) also has suggested on the basis of pilot balloon and radiosonde observations taken on a return voyage from Europe to South America that there are no solenoids to drive the trades.

This mass of conflicting evidence points out clearly that it is very dangerous to draw sweeping conclusions concerning the tropics as a whole from observations taken at a few stations and over short periods only. In view of the unsteady character of the upper flow, emphasized repeatedly, data such as Vuorela's hardly lend themselves to generalizations. It is necessary to gain a view on the tropical belt as a whole in order to obtain clarification. On account of scarcity of southern hemisphere data this is not yet possible. But the series of 300-mb charts for 1945, mentioned earlier, makes it possible to obtain at least a qualitative picture to about 10° N.

One monthly mean chart—that for August 1945—has been chosen for illustration (fig. 13). In August the zonal circulation in the north and the equatorward extent of the polar westerlies attain their seasonal minimum. Any tendency toward increase of east wind with height in low latitudes should be displayed most prominently. We note on fig. 13 that even though we are dealing with a monthly mean chart, marked contour patterns are nevertheless evident. In higher latitudes, a

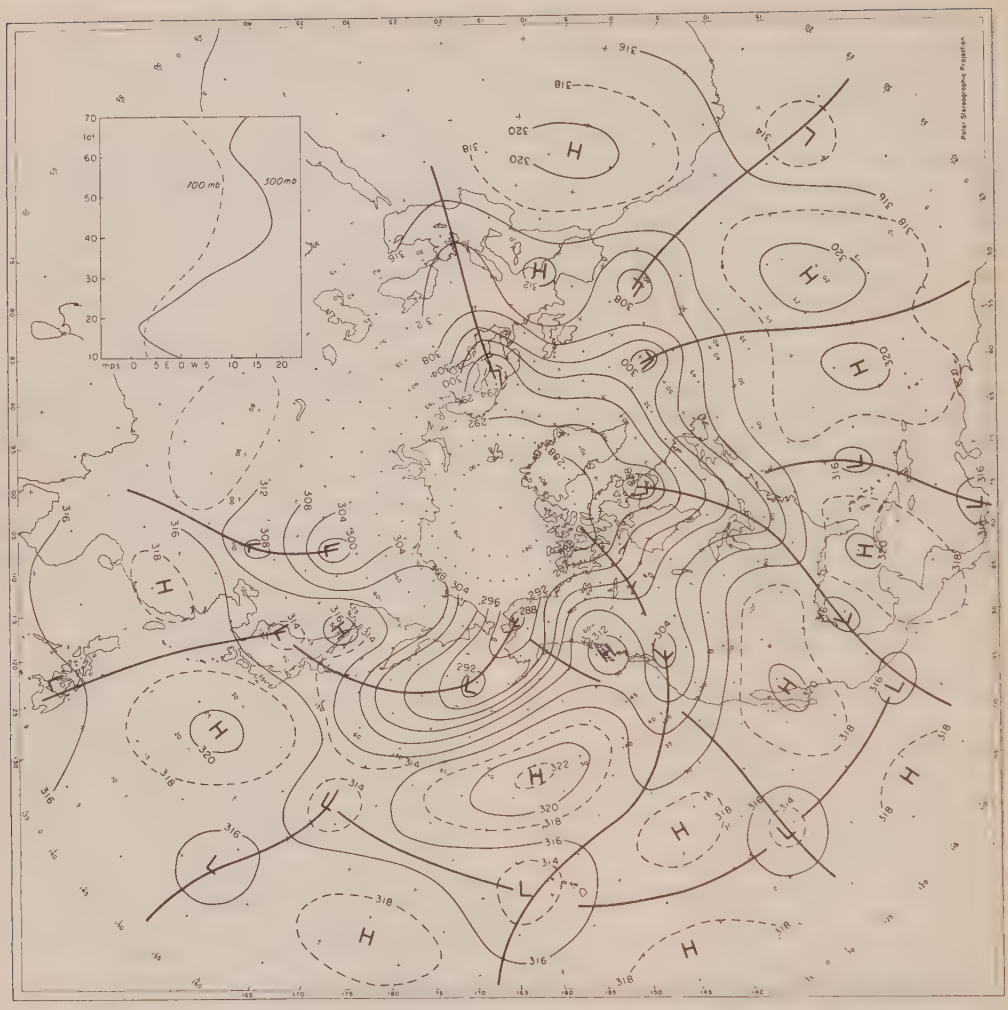


Fig. 13. Topography of the 300-mb surface (100's feet) for August 1945. This monthly chart was computed from 31 daily charts. Solid heavy lines are troughlines, dashed heavy lines mark the subtropical ridge. Insert: meridional profile of the zonal wind (mps) at 700 and 300 mb for August, 1945, computed for the area given in Fig. 2.

well-defined westerly current (jet stream) is in evidence. Pronounced troughs and ridges alternate indicating a wave number of four in the polar belt. The outstanding feature of the lower latitudes is the array of cellular subtropical highs with intermittent troughs, in good agreement with the cellular structure of the subtropics demanded by V. BJERKNES and COLL. (Physikalische Hydrodynamik, 1933, 679—687). *There is no zonal symmetry in the sense that all meridional planes show the same features.* Regions of northerly and southerly transport alternate. *A general antitrade is not present.*

All major troughs of high latitudes have counterparts in the tropics. Even on the monthly mean chart the existence of great extended troughs is the most impressive feature. It is evident, however, that the number of troughs in low latitudes exceeds that of high latitudes, in part presumably in correspondence to the much greater circumference of the earth. If in addition to the trough positions given by fig. 13 we avail ourselves of the knowledge that a very persistent upper-air trough lies near longitudes 75—80° E during summer, we obtain a mean hemispheric wave

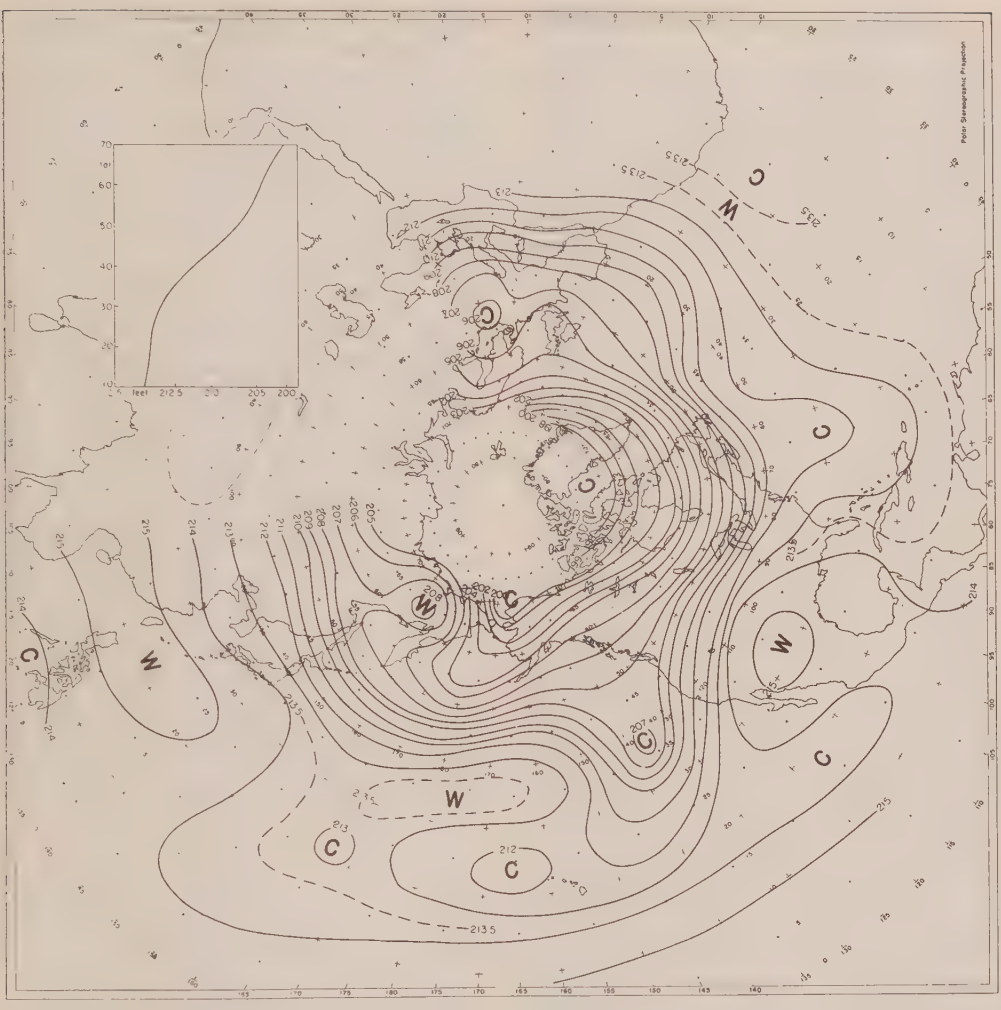


Fig. 14. Relative topography of the 300 above the 700-mb surface (100's feet) for August, 1945. "W." stands for warm and "C" for cold. Insert: meridional profile of relative topography (logarithmic scale) against latitude for the area indicated in Fig. 2.

number of seven or eight for the low latitudes. Perhaps the most remarkable asymmetry between high and low latitudes occurs in the eastern Pacific Ocean.

It is possible to compute the average zonal wind profile from fig. 13, if geostrophic conditions are assumed. Since the figure represents a monthly mean chart, this approximation presumably is valid even at low latitudes. Many observers have found that except in disturbances the deviations from balanced wind are smaller in the latitude range 10°—20° than previously assumed. Moreover, the geostrophic speeds as computed (insert of fig. 13) are in good agreement with those obtained from rawins in the tropics. Comparison of 300-mb and 700-mb shows *that the east wind decreases with height at all latitudes computed*, though it is probable from the 300-mb curve that the wind direction itself is east at that level at the equator. Even if the geostrophic assumption is only a rough approximation, the vertical change of wind must nevertheless be given correctly in the insert. Otherwise it is necessary to argue that there are systematic departures from geostrophic equilibrium that tend to produce supergeostrophic wind at 700-mb and subgeostrophic wind at 300-mb with

order of magnitude of 5 mps on the monthly chart.

It is possible, however, to dispense entirely with wind calculations and merely consider the solenoid field. This can be done qualitatively from the relative topography of the 300-mb surface above the 700-mb surface (fig. 14). Around the globe there are broad areas where the temperature gradient is directed northward at all latitudes. Interspersed are regions with reversed meridional temperature gradient. Fig. 14 perhaps brings out the lack of zonal symmetry most clearly. Notably in the central Pacific and over Mexico the east wind increases with height between latitudes 20° and 30°. The same holds true over the far western Pacific south of latitude 20°. This is also confirmed by rawins, especially over the Philippines and the Palau Islands group. *Thus, there is no single meridian that will yield a representative picture of the hemisphere.* Fig. 14 is in agreement with the cross-section of Hess (1948) since along the meridian 80° W the warmest air lies near 25° N. But the contour configuration over the western Caribbean is a local feature only. A little to the east, over the Antilles, the east wind must decrease upward, in accordance with the findings of Fassig (1933) and Stone (1942).

If the contours of fig. 14 are summarized for all latitudes, a profile results (insert of fig. 14) *that shows a northward directed temperature gradient at all latitudes.* This gradient is very small in the lowest latitude belt. Perhaps it is best to count it as zero there. As just seen, however, this does not mean that barotropy exists in any particular region. Figs. 13 and 14, together with all measured wind summaries aloft make it clear *that the average flow pattern as well as the daily chart is dominated by extended troughs and that therefore longitudinal asymmetry prevails. Regions of northward and southward directed temperature gradient alternate aloft as do regions of upper westwind and eastwind.*

Although fig. 14 demonstrates the presence of solenoids in low latitudes, it is still necessary to determine the energy source that maintains the windfield over the tropics. Sometimes it has been suggested that this energy may be derived from middle latitudes. Calculations of the heat energy released through condensation and precipitation in organized convection zones, however, show that only a negligible fraction (5 % or less) of the heat released need be converted into kinetic energy in order to account for the observed motion. This percentage is very small even in hurricanes (about 10 %). Thus there is no problem in obtaining energy to drive the currents of the tropics. The real problem is why there is so little motion, why the atmospheric heat engine is so inefficient.

Basic currents of the tropics

The structure of individual disturbances encountered in the variable broad-scale pattern must depend on the large-scale features themselves, especially the vertical variation of the zonal wind. The westerly current of higher latitudes almost always increases with elevation. Outside the tropics, therefore, a single type of basic current predominates. In contrast, the baroclinity of the easterlies is variable as just seen. Moreover, the tropics are subject to much larger seasonal changes of basic current structure than the higher latitudes. As a result, a greater variety of synoptic disturbances is encountered, and their occurrence has a definite seasonal course. In winter (generally dry season) the polar westerlies

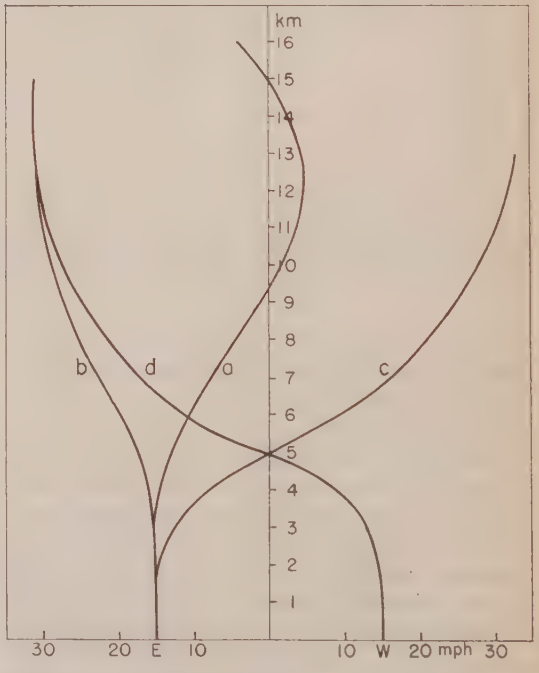


Fig. 15. Schematic representation of types of zonal currents typical of low latitudes.

aloft invade most tropical areas to at least latitude 10°. The base of these westerlies slopes equatorward with height from the latitude of the surface position of the subtropical ridgeline. Thus the depth of the trades is variable, ranging from a few hundred meters to almost the whole depth of the troposphere. Curve 'c' of fig. 15 illustrates this class of current. Curves 'a' and 'b' show the vertical variation of the trade as encountered in the rainy season (summer).

The picture is completed by introduction of the low-level "equatorial westerlies", a current that has remained least amenable to simple classification and understanding up to the present. Originally observed near the surface over India, the Far East, and West Africa, it was thought to be of southern hemisphere origin turning clockwise under the law of conservation of momentum as it crosses the equator in response to large-scale thermal differences between the hemispheres (monsoon). While this definition is not incorrect for certain areas, it is certainly incomplete. Surface westerlies in the equatorial zone not only blow toward heated continents, but also away from them to their east (RIEHL 1948). The classical type of equatorial west wind should be overlain by easterlies (curve 'd' of fig. 15). This is true in summer over India, but it is by no means observed with regularity in other regions. For such reasons it is best to avoid terminology as "southwest monsoon" that has a causal conotation. The term "equatorial westerlies", gives a definition that simply refers to the geographic fact that these westerlies are separated from the polar westerlies by an intervening wedge of easterlies. The interesting fact remains that we encounter few areas on earth where west winds do not occupy at least a portion of the troposphere.

The geographical arrangement of the four currents shown in fig. 15 follows a definite pattern. Near the poleward boundary of the tropics the trades give way to polar westerlies aloft. Low-level westerlies, overlain by easterlies, occur almost exclusively near the equatorial low pressure trough. The heat exchange between high and low latitudes of the same hemisphere largely depends on the vertical structure of the low-latitude windfield, since the latter has a definite relation to the meridional temperature gradient.[1] Poleward heat transport requires the presence of a northward directed temperature gradient, therefore a decrease of the easterlies with height. *Extended troughs form where this temperature field prevails.* As an example, comparison of fig. 13 and 14 shows that most of the principal extended troughs are situated in areas where the relative topography of the 300-mb surface decreases northward at all latitudes. In contrast, *interaction between high and low latitudes is shut off where the temperature gradient reverses.* There the easterlies increase upward to the tropopause level. *It is typical of these easterlies that they are very steady* (cf. RIEHL 1948, fig. 9). Heat energy realized from condensation presumably flows across the equator under these circumstances.

Another characteristic of the basic currents deserves mention. This is the striking opposition between lower and upper troposphere. As indicated in fig. 15, the motion that prevails near the ground differs greatly from that of the upper troposphere. A finite, generally narrow, transition zone extends between these regimes. This zone tends to be situated between 600 and 400 mb, though with many variations in individual situations. This observation points to the possibility of treating the tropical atmosphere as composed of two distinct layers in the vertical, with the 500-mb surface serving as dividing line.

As noted earlier, winds are very steady in the low levels while restlessness characterizes the high layers. There is an important exception to this statement. High-level easterlies that increase in intensity with height are very steady, as just mentioned, much more so than easterlies that weaken with height, or upper westerlies. In areas where westerlies occupy the low levels and are overlain by easterlies, eddy motion generally is strongest near the ground. The vertical gradient of wind steadiness then is reversed. It is noteworthy that a reversal of this quantity is always present. Pronounced eddies are present either above or below 500-mb. They rarely extend throughout the entire troposphere, in distinction to troughs and ridges of higher latitudes.

[1] This argument is not based on the thermal wind equation but on empirical evidence from maps containing rawin and temperature observations. The data show that the equation is not invalid as far as latitude 10°.

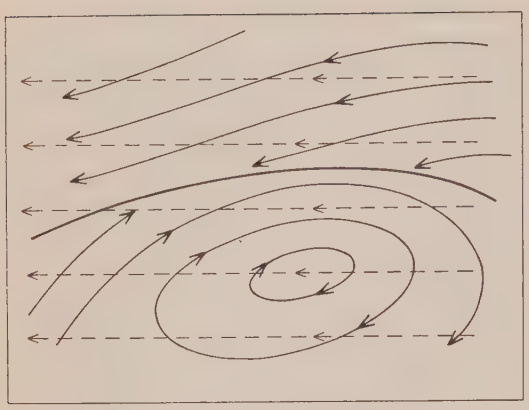


Fig. 16. Illustration showing uniform zonal current in one layer of troposphere (dashed lines) and equatorial shearline and eddy in other layer (solid lines).

Fig. 16 serves to illustrate this point. The pattern shown describes conditions frequently observed along the equatorial trough during the northern hemisphere summer. Two layers are depicted. A broad easterly current occupies one of these, while an intense shearline is located in the other. Winds are most unsteady along the shearline where cyclonic vortices develop, and to its south where motion of the clockwise eddy usually takes place. In fig. 2, conditions in the eastern and central Pacific most closely resemble this pattern. The astounding fact is that whereas fig. 16 is valid almost everywhere along the equatorial trough, it must be applied with alternating interpretation. In some regions, the shearline is located in the upper troposphere and in others at the ground (equatorial front or convergence zone). In many parts of the equatorial zone the layer of steady easterlies adjoins the surface. In others it prevails from 500-mb upward. As seen from figs. 10 and 11, however, the equatorial front zones are very narrow and occupy only a small fraction of the total tropical area.

In view of the foregoing, it is not surprising that synoptic studies report gross regional differences in equatorial weather patterns. If we recognize that a remarkable compensation exists between low and high levels, that upper and lower troposphere tend to have different and often quasi-independent flow patterns, and that the relation between these layers varies with longitude, it becomes possible to evolve a rational account of weather near the equator.

Conclusion

This report has attempted to demonstrate that the investigator of the general circulation cannot proceed like a physicist who turns on a well-controlled flame underneath a tank with fluid and then studies the motion in the tank without further reference to the flame. The source of heat for the middle latitude westerlies has a very variable character. Transfer of heat from the oceans surface to the high troposphere—the principal layer of transmittal to the polar zone—is locally and longitudinally not continuous. This also holds for poleward heat emission. Since local and longitudinal variations of heat exchange are in part determined by the tropical circulation itself, changes of the flow pattern at the receiving end in the temperate zone cannot be understood entirely without reference to the tropics. As yet, no attempt has been made in meteorology to evaluate dynamically these poleward directed influences.

Although heat accumulation and emission from the tropics is a function of longitude, and locally also of time, the question remains whether there are fluctuations of these quantities in time if we integrate around the entire tropical belt. The wartime extension of observational material has made possible studies of this important problem. Recent research, as yet uncompleted, has shown that the question can be answered in the affirmative.[1] The total internal energy of middle and upper troposphere in low latitudes, therewith the poleward heat transport, are subject to marked aperiodic fluctuations with a period of roughly two to three weeks. This observation makes it clear that the tropics must play an important part in the production of variations in the intensity of the circulation of middle latitudes.

[1] H. Riehl, T. C. Yeh, and N. LaSeur: A study of variations of the general circulation (to be published).

REFERENCES

Austin, J. M., 1948: A note on cumulus growth in a non-saturated environment. *J. Meteor.*, **5**. 103—107.

Bjerknes, J., 1938: Saturated-adiabatic ascent of air through a dry-adiabatically descending environment. *Quart. J. Roy. Met. Soc.*, **64**, 325—330.

Bjerknes, V., J. Bjerknes, H. Solberg, and T. Bergerons 1933: Physikalische Hydrodynamik. Berlin: Julius, Springer, 797 pp.

Brunt, D., 1939: Physical and dynamical meteorology. Cambridge University Press, 428 pp.

Bunker, A. F., B. Haurwitz, J. Malkus, A. H. Stommel, 1949: Vertical distribution of temperature and humidity over the Caribbean Sea. *Pap. Phys. Ocean. and Meteor., Mass. Inst of Tech. and Woods Hole Ocean. Inst.* **11**, No. 1, 82 pp.

Cressman, G. P., 1948: Studies of upper-air conditions in low latitudes. Part II: Relations between high- and low-latitude circulations. *Dep. Meteor. Univ. Chicago Misc. Rep.* #24, 68—100.

Defant, A., 1921: Die Zirkulation der Atmosphaere in den gemässigten Breiten der Erde. *Geogr. Ann.*, **3**, 209.

Fassig, O. L., 1933: The trade winds of the eastern Caribbean. *Trans. Amer. Geoph. Union*, **14**, 69—78.

Hess, S. L., 1948: Some new meridional cross sections through the atmosphere. *J. Meteor.*, **5**, 293—300.

Hubert, L. F., 1949: High tropospheric westerlies of the equatorial west Pacific Ocean. *J. Meteor* **6**, 216—224.

Jacobs, W. C., 1942: On the energy exchange between sea and atmosphere. *J. Marine Res.*, **5**, 37—66.

Kuhlbrodt, E., 1928: Das Strömungssystem der Luft über dem tropischen Atlantischen Ozean nach den Höhenwindmessungen der Meteor-Expedition. *Z Geophysik*, **4**, 385—386.

Langwell, P. A., 1948: Inhomogeneities of turbulence, temperature and moisture in the West Indies trade region. *J. Meteor.*, **5**, 243—246.

Priestley, H. C. B. and W. C. Swinbank, 1947: Vertical transport of heat by turbulence in the atmosphere. *Proc. Roy. Soc. A.*, **189**, 543—561.

Riehl, H., 1945: Waves in the easterlies and the polar front in the tropics. *Dep. Meteor., Univ. Chicago Misc. Rep.* #17, 79 pp.

— and E. Schacht, 1947: Some aspects of Puerto Rican rainfall. *Trans. Amer. Geoph. Union*, **28**, 401—406.

— 1948: On the formation of typhoons. *J. Meteor.*, **6**, 247—264.

— 1949: Some aspects of Hawaiian rainfall. *Bull. Amer. Met. Soc.*, **30**, 176—187.

Rossby, C.-G., 1949: On the nature of the general circulation of the lower atmosphere. The atmospheres of the earth and planets, edited by G. P. Kuiper. University of Chicago Press, 16—48.

Stommel, H., 1947: Entrainment of air into a cumulus cloud. *J. Meteor.*, **4**, 91—94.

Stone, R. G., 1942: On the mean circulation of the atmosphere over the Caribean. *Bull. Amer. Met. Soc.*, **23**, 4—16.

Vuorela, L. A., 1948: Contribution to the aerology of the tropical Atlantic. *J. Meteor.*, **5**, 115—117.

Woodcock, A. H. and J. Wyman, 1946: Convective motion in air over the sea. *Annals N. Y. Acad. Sci.*, **48**, 749—775.

On the Sensitivity of the Wind Field Relative to Pressure Variations

By SVERRE PETTERSSEN, U.S. Air Force

(Manuscript received 27 October 1949)

Abstract

It is assumed that the motion is frictionless and adiabatic but not necessarily horizontal. A geostrophic approximation of second order is introduced, and a measure of the sensitivity of the wind field relative to pressure variations is obtained. It is shown that the sensitivity depends essentially upon the lateral shear of the geostrophic wind and the tangential and orthogonal curvature of the geostrophic streamlines. In particular, when the geostrophic wind is large, the effect of the orthogonal curvature is predominant.

1. *Notations.* — The following symbols will be used.

$V = (u, v, w)$ = wind velocity.
$\dot{V} = (\dot{u}, \dot{v}, \dot{w})$ = acceleration.
$\ddot{V} = (\ddot{u}, \ddot{v}, \ddot{w})$ = derivative of acceleration.
$V_g = (u_g, v_g)$ = geostrophic wind.
K_i = tangential curvature of geostrophic streamlines, positive or negative according as the curvature is cyclonic or anticyclonic.
K_n = orthogonal curvature of geostrophic streamlines, positive or negative according as the streamlines converge or diverge.
E = mean radius of the figure of the earth.
g = acceleration of gravity.
Φ = angle of latitude.
Ω = angular speed of the earth's rotation.
$f = 2\Omega \sin \Phi$, $f' = 2\Omega \cos \Phi$
p = atmospheric pressure.
α = specific volume.
T = temperature.
c_p = specific heat at constant pressure.
Z_Θ = height above the reference level of an isentropic surface.
ψ = the Montgomery acceleration potential for an isentropic surface (1937).

$$\psi = C_p T + g Z_\Theta$$

In particular, if the y-axis is chosen along the direction of the horizontal pressure force

$$\frac{\partial \psi}{\partial x} = 0, \; v_g = 0, \; u_g = V_g = -\frac{1}{f}\frac{\partial \psi}{\partial y} \qquad (2)$$

and also

$$\frac{\partial^2 \psi}{\partial x^2} = -K_i \frac{\partial \psi}{\partial y},$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = K_n \frac{\partial \psi}{\partial y}, \; \frac{\partial^2 \psi}{\partial y^2} = -f\frac{\partial V_g}{\partial y} \qquad (3)$$

Furthermore, the following auxiliary notations will be introduced

$$I_x = -\frac{1}{f^2}\frac{\partial^2 \psi}{\partial x \partial t}, \; I_y = -\frac{1}{f^2}\frac{\partial^2 \psi}{\partial y \partial t} \qquad (4)$$

and may be identified with the components of the so-called isallobaric wind, in the meaning defined by BRUNT and DOUGLAS (1928).

2. *The Equations of Motion.* — Choosing the x-axis toward the east and the y-axis toward the north, the equations of frictionless motion may be written

$$\dot{u} - fv + f'w = X$$
$$\dot{v} + fu \qquad = Y$$

where X and Y represent the horizontal components of the pressure force.

Differentiating the above equations with respect to time and eliminating $\dot{u}$ and $\dot{v}$, we obtain

$$\left.\begin{aligned} \ddot{u} + f^2 u - v\dot{f} + w\dot{f}' + f'\dot{w} &= \dot{X} + fY \\ \ddot{v} + f^2 v + u\dot{f} + wff' &= \dot{Y} - fX \end{aligned}\right\} \quad (5)$$

Since no further differentiation will be performed we may leave out of consideration certain small terms the magnitudes of which do not exceed about one or two percent of the major terms.

Now, $\dot{f} = vf'/E$, and $\dot{f}' = -vf/E$. Except near the equator and near the poles, f and f' are about equal and of the order of 10^{-4} sec.$^{-1}$, while E is $6.4 \cdot 10^6$ m. In large scale currents, u and v are of the order 10 m/sec., w of the order 10^{-1} m/sec., and $\dot{w} = 10^{-5}$ m/sec.^{2} or less. It follows then that the magnitude of the terms $(\dot{f}v - \dot{f}'w - f'\dot{w})$ and $(\dot{f}u + ff'w)$ are about one or two percent of those of the terms f^2u and f^2v, respectively, and may therefore be omitted. The differentiated equations of motion, therefore, reduce to

$$\left.\begin{aligned} \ddot{u} + f^2 u &= \dot{X} + fY \\ \ddot{v} + f^2 v &= \dot{Y} - fX \end{aligned}\right\} \quad (6)$$

We may now choose the system of coordinates such that the positive y-axis coincides with the horizontal pressure force, for the only terms in (5) affected by this rotation are those involving f', which have been shown to be negligible. With this choice of coordinate axes, the horizontal pressure force may be expressed in either of the following forms

$$\left.\begin{aligned} X &= -\frac{\partial \psi}{\partial x} = -fv_g = 0 \\ Y &= -\frac{\partial \psi}{\partial y} = fu_g = fV_g \end{aligned}\right\} \quad (7)$$

Now, if the motion is adiabatic, then without assuming horizontal motion

$$\left.\begin{aligned} \dot{X} &= -\frac{\partial^2 \psi}{\partial x \partial t} - \frac{\partial^2 \psi}{\partial x^2} u - \frac{\partial^2 \psi}{\partial x \partial y} v \\ \dot{Y} &= -\frac{\partial^2 \psi}{\partial y \partial t} - \frac{\partial^2 \psi}{\partial x \partial y} u - \frac{\partial^2 \psi}{\partial y^2} v \end{aligned}\right\} \quad (8)$$

Substituting from equations (7) and (8) in (6), and using the notations defined in section 1, the differentiated equations of motion may be written

$$\left.\begin{aligned} \ddot{u} &= Au + Bv + E \\ \ddot{v} &= Cu + Dv + F \end{aligned}\right\} \quad (9)$$

where the coefficients have the following meaning:

$$A = -f(f + V_g K_i); \qquad B - C = fV_g K_n;$$

$$D = -f\left(f - \frac{\partial V_g}{\partial y}\right); \qquad E = f^2 (V_g + I_x);$$

$$F = f^2 I_y$$

If the corresponding coefficients are computed for the isobars in a level surface, additional terms appear, expressing the effect of the isosteric-isobaric solenoids. Except in frontal regions, where the degree of baroclinity is appreciable, the difference between the isentropic and isobaric coefficients is small, and for most purposes the two sets may be used interchangeably.

From a formal point of view, equations (9) are particularly well suited for analyses of the oscillations and stability of air currents. In the first approximation one may then assume that A, B, C, and D (which are second space derivatives of the pressure force) are constants. However, the coefficients E and F ($F = v_g + I_y$, with $v_g = 0$) are essentially first space derivatives of the pressure force, and in order to obtain solutions to (9) it is necessary to introduce assumptions as to the relation betwen E, F and u, v. Since any such assumption, capable of a clear physical interpretation, is difficult to formulate in a manner consistent with a complete set of equations, solutions to (9) will not be sought. Instead, an assumption will be made concerning the acceleration, so as to obtain an approximation to the instantaneous relation between the wind in a general pressure field.

3. *Geostrophic Approximation of Second Order.* — It is well known that the assumption of geostrophic balance gives a fair approximation to the true wind in the large-scale pressure systems. It is then assumed that the acceleration is small in comparison with the Coriolis acceleration, viz.,

$$|\dot{V}| \ll |2\Omega \times V| \quad (10)$$

If this is to hold not only at a given moment but all the time, it is reasonable to assume that

$$|\ddot{V}| \ll \left| \frac{d}{dt} (2\,\Omega \times V) H \right| \qquad (11)$$

We shall refer to (10) and (11) as geostrophic approximations of first and second order, respectively. Since neither of the above assumptions can be justified by reference to mathematical argument, their usefulness must be determined by comparison of derived results with observation.

In the following it will be assumed that the motion is sufficiently geostrophic for the terms containing second derivatives of the velocity to be neglected in comparison with the derivative of the Coriolis force.

4. *Sensitivity of the Wind Field.* — Returning now to equation (9) and introducing the second order geostrophic approximation, we obtain (since $B = C$) the following expressions for the velocity components

$$\left.\begin{aligned} u &= \frac{CF - DE}{AD - C^2} \\ v &= \frac{CE - AF}{AD - C^2} \end{aligned}\right\} \qquad (12)$$

Now, since the x-axis is chosen along the geostrophic wind, v is the cross-isobar component of the actual wind. In balanced motion this component vanishes (i.e., $CE = AF$). However, for any value of $(CE - AF)$ different from zero, the cross-isobar component and the tangential acceleration increases as the denominator decreases. Hence, with some justification, the quantity

$$S = \frac{1}{AD - C^2} = \frac{1}{\left(1 + \frac{1}{f} V_g K_i\right)\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right) - \frac{1}{f^2} V_g^2 K_n^2} \qquad (13)$$

may be taken as a measure of the sensitivity with which the wind reacts relative to pressure variations.

It should be emphasized, however, that (12) and (13) represent an approximation, the accuracy of which depends upon the smallness of $\ddot{u}$ and $\ddot{v}$ as compared with uf^2 and vf^2, respectively.

It is of interest to note that the above expression for the sensitivity of the wind field is related (inversely) to the degree of dynamic stability. Solberg (1936) considered a zonal geostrophic current (i. e. $K_i = K_n = 0$) and found the degree of dynamic stability expressed by $\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right)$, which agrees with (13). Van Mieghem (1944) considered a symmetrical circular vortex (i. e., $K_n = 0$) and found the degree of stability expressed by

$$1 - \frac{1}{f}\frac{\partial V_g}{\partial y} + \frac{1}{f} V_g K_i$$

which (on the same assumptions) differs from (13) by the absence of the quadratic term $\frac{1}{f^2} V_g K_i \frac{\partial V_g}{\partial y}$.

5. *Types of Pressure Fields.* — Substituting the definitions of the coefficients A, B, etc. given with equation (9) into (12), we obtain for the velocity components

$$\left.\begin{aligned} u &= \frac{(V_g + I_x)\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right) + I_y \frac{1}{f} V_g K_n}{\left(1 + \frac{1}{f} V_g K_i\right)\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right) - \frac{1}{f^2} V_g^2 K_n^2} \\ v &= \frac{I_y\left(1 + \frac{1}{f} V_g K_i\right) - (V_g + I_x)\frac{1}{f} V_g K_n}{\left(1 + \frac{1}{f} V_g K_i\right)\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right) - \frac{1}{f^2} V_g^2 K_n^2} \end{aligned}\right\} \qquad (14)$$

It will be seen that the deviation of the wind from the geostrophic wind depends partly upon the isallobaric gradient (here represented by I_x and I_y), and partly upon the lateral shear of the geostrophic wind and the tangential and orthogonal curvatures of the streamlines of the geostrophic wind (i. e. the curvatures of the lines p = constant in the barotropic case, and ψ = constant in the baroclinic case).

(1). *Stationary geostrophic wind.* — In this case $I_x = I_y = 0$, and the angle (β) of deflection of the actual wind from the geostrophic wind is given by

$$\tan \beta = \frac{V_g K_n}{f - \dfrac{\partial V_g}{\partial y}} \qquad (15)$$

As will be seen from Table 1, the magnitude of β may be appreciable. At such low geostrophic velocities as occur near the earth's surface, β is normally less than the frictional deflection and is, therefore, difficult to detect. In the upper troposphere in middle latitudes where geostrophic winds in excess of 50 m/sec. are a normal occurrence, very large angles of deflection will occur unless the isobars are almost strictly parallel.

Table 1

Magnitude of the Deviation of the Direction of the Actual Wind from that of the Geostrophic Wind (in degrees) for Various Values of V_g (in m/sec.) and K_n (in km) in Latitude 45°, computed from Eq. 15.

	$\frac{\partial V_g}{\partial y} = \frac{1}{2} f$			$\frac{\partial V_g}{\partial y} = 0$			$\frac{\partial V_g}{\partial y} = -f$		
V_g / $\pm K_n$[1]	10	40	80	10	40	80	10	40	80
400	27	65	76	14	45	65	7	27	45
800	14	45	65	7	27	45	3	14	27
1 600	7	27	45	3	14	27	2	7	14

It will be seen from (13) that the sensitivity decreases as $V_g^2 K_n^2$ increases.[1] Hence, the strong winds present in the upper part of the troposphere are extremely sensitive relative to isobar convergence and divergence. It is often observed that appreciable developments in the pressure field occur at and below regions of such convergence in the upper troposphere. Normally, appreciable isobar convergence, or divergence, associated with strong winds result in pressure variations, with the result that the angle β is modified by the isallobaric gradient. The figures in Table 1 suffice, however, to demonstrate the importance of K_n as far as the sensitivity of the wind field is concerned. Since the acceleration tangential to the isobar is equal to $fv \equiv fu \tan \beta$, it follows that regions of appreciable isobar convergence or divergence are characterized by large accelerations.[2]

(2). *Straight and parallel isobars without geostrophic shear.* — In this case $K_i = K_n = \partial V_g / \partial y = 0$, and (14) reduce to

$$u = V_g + I_x, \quad v = I_y \qquad (16)$$

which is the well-known BRUNT-DOUGLAS equation (1928), showing that for this type of pressure field the actual wind is equal to the geostrophic wind supplemented by a vector which is proportional to the isallobaric gradient. It will be seen that in this simple type of pressure field $S = 1$.

(3). *Straight and parallel isobars with geostrophic shear.* — In this case (14) reduce to

$$u = V_g + I_x, \quad v = I_y \Big/ \left(- \frac{1}{f} \frac{\partial V_g}{\partial y} \right) \qquad (17)$$

It will be seen that the component along the isobar is the same as in the foregoing case, while the cross-isobar component is larger or smaller according as the geostrophic shear is anticyclonic (> 0) or cyclonic (< 0). It follows then that *the wind field is more sensitive to pressure changes when the geostrophic shear is anticyclonic than it is when the shear is cyclonic.*

(4). *Curved concentric isobars.* — In this case $K_n = 0$, and (14) reduce to

$$\left.\begin{aligned} u &= (V_g + I_x) \Big/ \left(1 + \frac{1}{f} V_g K_i \right); \\ v &= I_y \Big/ \left(1 - \frac{1}{f} \frac{\partial V_g}{\partial y} \right) \end{aligned}\right\} \qquad (18)$$

and the accelerational components are

$$u = \frac{f I_y}{1 - \dfrac{1}{f} \dfrac{\partial V_g}{\partial y}}; \qquad v = \frac{V_g^2 K_i - f I_x}{1 + \dfrac{1}{f} V_g K_i} \qquad (19)$$

It will be seen that *the motion is more sensitive to pressure variations when the geostrophic shear and curvature are anticyclonic than when they are cyclonic.*

[1] The abnormal case, when either of the two parentheses in (13) is negative, will not be considered since the wind could not then be predominantly geostrophic.

[2] SCHERHAG (1948), who places much emphasis on the isobar convergence and divergence, interprets these features as convergence and divergence of the actual motion.

The relation between the wind and the above-mentioned simple pressure fields is shown diagrammatically in fig. 1. In diagram *A*, there is no geostrophic shear and the geostrophic departure is determined exclusively by the isallobaric gradient, according to BRUNT and DOUGLAS equation. In diagram *B* the geostrophic departure is amplified through anticyclonic shear, and in *C* it is reduced through cyclonic shear. Diagrams *D* and *E* show the conditions around a cyclone and an anticyclone moving from the left to the right. Here, the effects of shear and curvature combine to give a large variety of geostrophic departures. It will be seen that the winds are rather insensitive to pressure changes along the periphery of the anticyclone and in the central part of the cyclone, while the reserve is true along the periphery of the cyclone and near the centre of the anticyclone.

(5). *Converging and diverging isobars.* — In this case the relation between the wind and the pressure field is given by (14) and since both u and v depend upon all the characteristics of the pressure field, the relationship is not readily represented in a simple diagram. It may be noted, however, that the cross-isobar components of the wind are determined primarily by the isallobaric gradient and the orthogonal curvature. These components are, however, amplified or reduced according as the sensitivity factor is larger or smaller than unity. It will be seen from (13) that the effect of isobar convergence or divergence (i. e., K_n) is always to increase the sensitivity factor. Particularly in the upper part of the troposphere in middle latitudes, where the geostrophic winds are very strong, the sensitivity of the wind field may be greatly increased

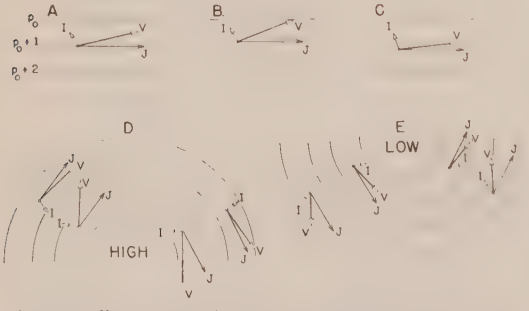


Fig. 1. Illustrating the relation between the actual wind (V), the geostrophic wind (J) and the isallobaric wind in relation to isobar shear and curvature.

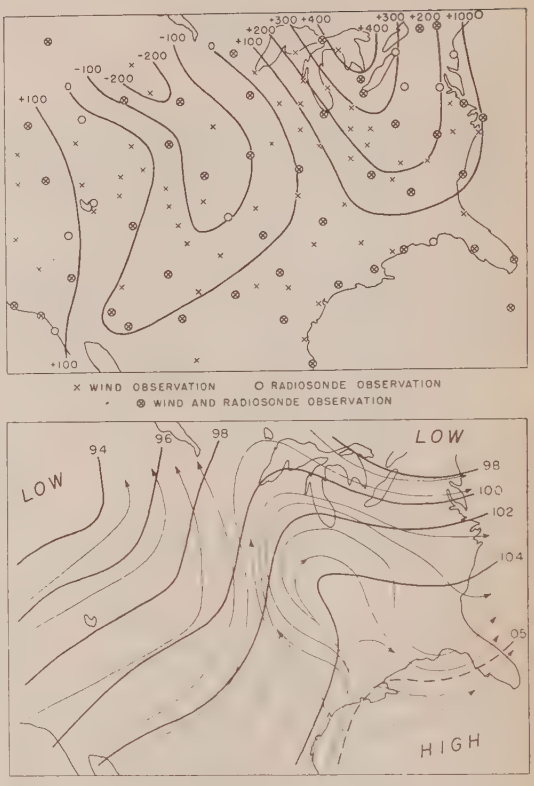


Fig 2. Upper chart: 12-hour tendency of the height of the 700 mb. surface, April 3, 1948, 0930 to 2130 G. M. T.
Lower chart: streamlines of the actual wind and contours of the 700 mb. surface, April 3, 1948, 1530 G. M. T.

even through moderate amount of isobar convergence and divergence.

Figure 2 shows, as an example[1], the streamlines of the actual wind in relation to the pressure force and the isallobaric field. An examination of the charts will show that eq. (14) is capable of accounting for all essential features of the deviation of the actual wind from the geostrophic wind.

The extent to which the sensitivity factor (eq. 13) will prove useful as a synoptic indication of development in the pressure field remains to be determined. It is hoped that a synoptic investigation by A. F. GUSTAFSON and L. C. STARRETT, which will be completed shortly, will clarify certain aspects of this problem.

[1] The streamline chart is reproduced with the kind permission of Major A. F. GUSTAFSON, U.S. Air Force, from his unpublished report on a detailed analysis of the wind and pressure fields over the USA during the International Aerological Days April 1948.

REFERENCES.

BRUNT, D. and DOUGLAS, C. K. M. 1928: On the modification of the strophic balance; *Mem. Roy. Met. Soc.* **3**, 22.

MONTGOMERY, R. B. 1937: A suggested method for representing gradient flow in isentropic surfaces. *Bull. Am. Met. Soc.* **18**. pp. 10—12.

SOLBERG, H. 1936: Le mouvement d'inertie de l'atmoshpère stable et son rôle dans la théorie des cyclones. *Mem. Met. Ass. U. G. G. I.*

VAN MIEGHEM, J. 1944: Perturbation d'un courant atmosphérique zonal. *Classe de Science de l'Ac. de Belgique.*

Duration of Lightning Strokes and Occurrence of Multiple Strokes

By HARALD NORINDER
Institute of High Tension Research, University of Uppsala

(Manuscript received 10 December 1949)

Abstract

It is shown that the total duration of lightning strokes, the occurrence of multiple strokes and the time interval between them are partially influenced when measurements are made on direct strokes to tall objects. By recording the radiated electromagnetic field of lightning discharges in antenna circuits with cathode ray oscillographs at great distances from the lightning paths it was possible to obtain extensive data on the variation characteristics of lightning. Special methods were developed in order to analyse the lightning discharges with regard to the sequence of multiple strokes and the slow moving after-variation. A comparison of the discharge characteristics of lightning in Sweden with corresponding results from lower latitudes leads to the assumption that the intensity of thunderstorms is greater in low latitudes than in high latitudes. This involves researches of a geophysical problem of greatest interest from different points of view.

Introduction

During the decade before the second great war researches were carried out in several countries of the electrophysical conditions of lightning discharges. In many cases it was industrial electrotechnical enterprises which were responsible for this research work. Especially was this the case in the United States of America. As a matter of course the researches were to some extent carried out with reference to protective applications — hence directed on certain practical problems of lightning discharges. This did not prevent a more general treatment of the problem, which may be exemplified by the contents of a comprehensive article by WAGNER and MC CANN [1], entitled Lightning Phenomena. This article, which is of great interest from geophysical points of view, deals with results up to 1944.

As a result of the investigations described the authors consider certain characteristics of lightning that may be said to be quite definitely known. On the other hand, others are considered to be only partially answered.

To the first group they consider as solved the mechanism of the discharge, its polarity and its wave or impulse type. To the same group of solved problems they count the total duration of lightning discharges, the number of discharges in the lightning path and the time interval between them. Especially the three last mentioned characteristics of lightning discharges are of the greatest importance with regard to protective devices against lightning strokes. On the other hand, they are of the greatest interest from a purely electrophysical point of view.

Against the quoted authors' opinion that the last mentioned characteristics of lightning discharges have reached a definite and conclusive solution, I am of the opinion that the researches upon which the conclusions are founded must not be considered as definite.

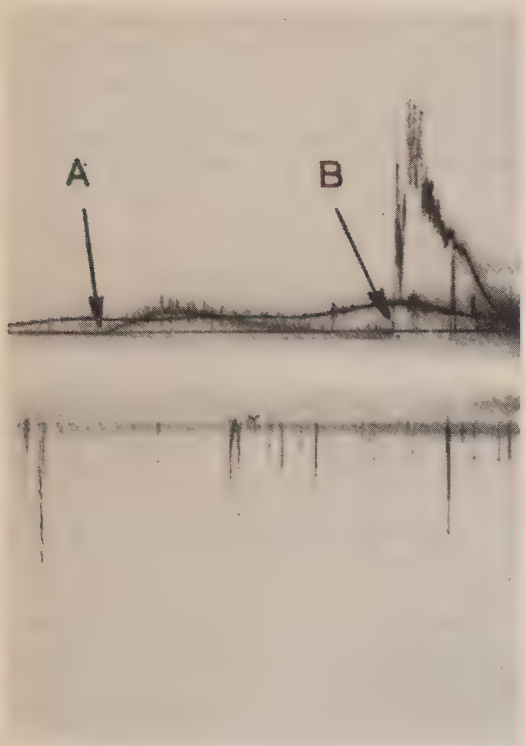


Fig. 1. Original cathode ray oscillogram of lightning discharge taken at the Uppsala Research Laboratory twenty years ago. The figure shows predischarges A—B and a steep, fronted main discharge with following multiple strokes.

The researches have not been carried out during such general conditions that we will be able to consider the results as valid for general discharges in open country topographical conditions. The reasons for this are evident. To a great extent the quoted results have been obtained through the study of lightning strokes to tall structures. In other cases the discharge intervals have been determined by photographic methods without any consideration of the influence of the after glowing by ionization in the lightning path. In as much as the equivalence between the discharge time interval as carried out by photographic methods, and the corresponding measurement by electromagnetic methods has not been proved, we must critically consider the former method. The weakest point of the results hitherto gained must be, as has already been pointed out, that the measurements have been made on strokes to tall structures. They produce very pronounced and local disturbances of the electromagnetic field from which must follow lightning discharge variations typical for such a structure but not for open field conditions. The above mentioned authors themselves remark that the quoted results from tall structures are debatable from the considered point of view. The tall structure exposed in the field of a thunderstorm cloud will cause that a greater volume of the cloud will be ionized as compared with field conditions on open and flat ground and hence cause a heavier current in the lightning paths. This involves both a greater number of partial discharges and increased duration. This is evident the more as some of the discussed measurements have been obtained by using discharges to an earth-connected captive balloon at heights of 500 to 800 metres which must introduce very pronounced local disturbances of the electromagnetic field. Similar disturbed field conditions even more difficult to overlook follow from results obtained by measuring lightning strokes to a sharp pointed peak of a mountain. Graphs illustrating the discussed results will be quoted further on.

Considering these limitations of the method of measuring lightning strokes to tall structures it is astonishing that a more safe and convenient method to measure lightning discharges has not been extensively tried. Such a method is at hand by using the radiation of the electromagnetic field from lightning discharges. In this case it is possible to operate with open or closed antenna circuit systems — a method that has been extensively applied at the Institute of High Tension Research. For further information about the principles of methods see the authors' article in this journal, vol. 1, no. 2, [2] and NORINDER-DAHLE [3].

The General Aspects of the Lightning Discharges as Investigated with Cathode Ray Oscillographs

Before going into the details of the variation features of lightning discharges it will be necessary to consider the three characteristic stages of the lightning discharges: a) the initial — or the predischarges with a total duration of up to 20.000 μsec.; b) the rapid main and partial discharges with a total duration of up to 2.000 μsec.; c) the slow moving after-

variation process with a total duration of about 10.000 μsec. or occasionally longer. Very often we have observed that the total duration is shorter than the values mentioned above. With the mentioned values at hand we have to estimate the total duration of a lightning stroke procedure to be of the order of up to 0.03 second. It is not easy to record with an oscillograph these long time variation characteristics of a lightning discharge with a satisfactory resolving of the details. During our experiments in Sweden, started nearly thirty years ago, we used an open antenna circuit in combination with a cathode ray oscillograph with a rotating drum arrangement inside the vacuum tube. At an early stage of our researches these records of the variation feature of lightning discharges enabled us to discover the existence of the predischarges, their amplitudes and their duration. From the same records it was also possible to verify in a more definite manner our earlier discovery that the lightning discharges were aperiodic. Earlier it had been claimed that they were oscillating of a high frequency and sometimes as high values as 10^6 periods were supposed to be characteristic of a lightning discharge. In fig. 1 we reproduce an example of such an old record where the visible predischarges are marked between A—B.

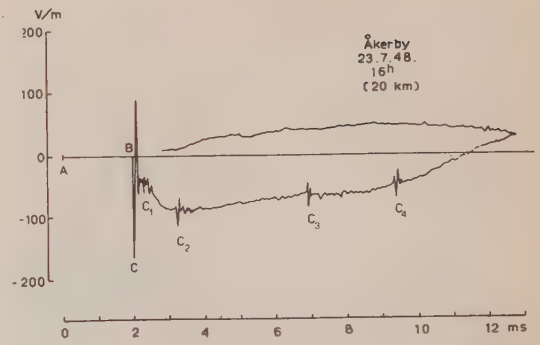


Fig. 2. Oscillogram of lightning discharge taken at one of the Institute's field station with recently developed recording device for long durations. Predischarges between A—B, main discharge at C followed by partial discharges from C_1 to C_4. Long after-variation.

Experimental Considerations when Measuring Variations of Lightning Discharges with New Arrangements

We were during these earlier experiments constrained to operate the drum in vacuum during a certain time period which necessitated that several consecutive discharges had to be recorded on the same oscillogram. This must cause a certain merging of the variation curves of the different discharges. This earlier problem of analysing the total variation traits of lightning discharges seemed of different reasons to be of important value. Hence quite recently a special arrangement was introduced in our cathode ray oscillographs in order to take up these older investigations in a modernized way. The cathode ray oscillographs were provided with a time base variation sweep of an axial length of 20 centimetres. With such a linear time circuit, sweeping several times above the long time axis, it was possible to obtain detailed records covering the whole discharge period of lightning. The relay arrangement of the author's cathode ray oscillograph construction (ACKERMANN [4]) prevented a recording when the cathode ray oscillograph was passing the regions of zero. This part of the recording is sometimes interpolated by dotted lines in the following curves of lightning discharges. The oscillograph, when operated with the long time axis, was connected to an open antenna circuit and the observations were taken with the E method which thus resulted in direct records of the field force in volts/meter. (See ref. [2]) Some results of such investigations are reproduced in fig. 2—6. The distances between the lightning paths and the observation stations vary between 18 to 40 km, and when possible to observe, these distances are given (in km) in the figures. A control that local influences did not affect the results was introduced by recording the discharges simultaneously on two stations at a distance of 18 km between them. This arrangement also made it occasionally possible to record the same lightning discharge at the two stations. As will be observed later on, the amplitude of the electric force of the predischarges of a lightning stroke is much smaller than the corresponding values of the main discharge. Hence it is not always possible to obtain a balance value in lightning discharges of the amplitudes mentioned such as can be recorded in full with a voltage sensibility of the oscillographs suitably fitted to one of the two amplitudes. Field variation curves drawn from

an oscillogram taken with the method mentioned above at one of the field station of the Institute, Åkerby, at a distance from the lightning path of 20 km, is reproduced in fig. 2. The oscillogram does not show the predischarges, because they were too faintly

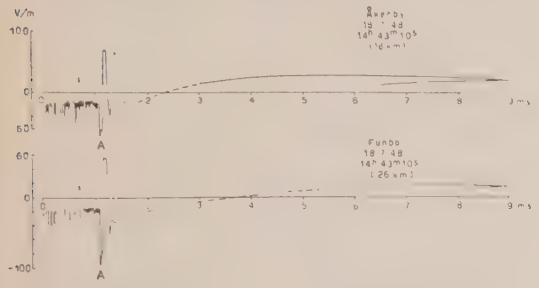


Fig. 3. Simultaneous records of lightning discharge from two field stations. Typical recorded predischarges before main discharge A. No partial discharges are following.

recorded. The predischarges were thus estimated to have a duration of 2 msec (milliseconds) marked A—B fig. 2. The main and partial discharges and the slow moving after-variation are reproduced from the original oscillogram. The predischarges are followed by the rapid main discharge C and first partial discharge at C_1. On the slow moving after-variation superimposed traces of subsequently occurring partial discharges are marked as C_2—C_4. The discharges in the lightning path which are carrying the heavy currents are evidently the ones marked with C and C_2 to C_4. Most frequently it is so that the greatest amplitudes are reached by the heavy current carrying discharges in the *first* lightning stroke. Sometimes we observe exceptions from this rule. See fig. 3, where simultaneous records from the two stations at a distance of 18 km apart are drawn. The typical predischarges with a duration of up to 1.1 msec are visible. The lightning is in this case characterized by only one main discharge marked with A and followed by a slow moving after-variation.

In fig. 4 oscillograms of three separate lightning paths with unipolar variation type are reproduced. We observe how variable are the durations of the partial discharges or multiple strokes. In A_1 and A_3 all partial discharges are concentrated within a time period shorter than a millisecond and in A_2 they extend over more than two milliseconds.

Oscillograms of two lightning paths ocurring within a short time interval and recorded on the two stations and characterized by multiple strokes are reproduced in fig. 5. How shifting the variation type of a lightning may be, is shown by a comparison of the simultaneous records of the oscillograms of the two stations in fig. 6 with the foregoing curves. In this, as in the foregoing cases with exception of fig. 3, the predischarges were developed with such a low amplitude that they failed to be recorded within the sensibility of the oscillographs used. Hence the time is marked to begin with the first main discharge A_1, which is followed by 6 discharges up to A_7, of which the third attains the highest amplitude. Later on some additional discharges are visible superimposed on the slow moving after-variation.

The typical after-variations of sometimes very long duration are caused by rapid dislocations of charges within those regions of the thunderstorm cloud which are taking part in the charge transportation by the main and the partial discharges. Hence this procedure is provoking slow moving after-variations within the regions of the thunderstorm cloud which are active in the transportation of charges by the lightning. If we suppose the transported charges to be distributed within homogeneous

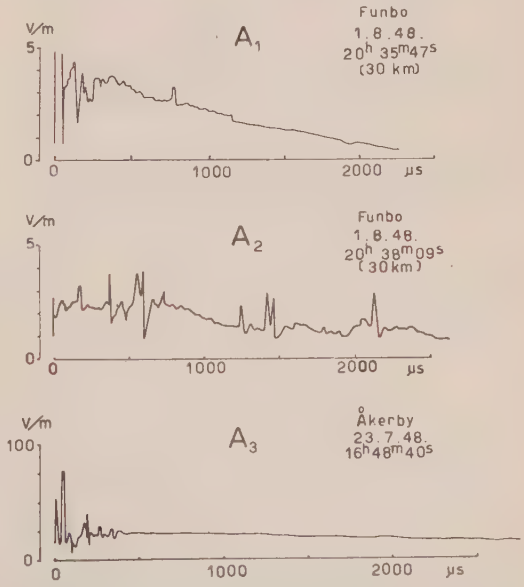


Fig. 4. Typical unipolar lightning discharges recorded from the Institute's field stations. Limited discharge time.

volumes of a sphere, a cylinder or a cone, it is rather easy to calculate the field changes and their polarity. The above mentioned variations of the thunderstorm fields have all been measured above open, flat extended surfaces. Thus local disturbances in the electric field conditions are eliminated.

If we, on the other hand, consider the corresponding variations when we are dealing with the field conditions in the vicinity of tall structures, other field variations are developed. As a prominent influence of the intensified field conditions in the vicinity of the tall structures, we have to expect both a lengthening of the time variation of the slow moving after-variation in the lightning discharge and also an intensification of the field charges. This will result in increasing values of the currents in main and partial discharges.

Results from Variations of Magnetic Field as Produced by Lightning Discharges

A typical consequence of such extended discharge periods in the lightning paths must be that transported electric charges, in the lightning attain very high values of the order of 100 coulombs or more. On the other hand, the author's measurements in open field conditions by using the electromagnetic frame aerial method (ref. [3]) resulted in values of up to 10 coulombs and only in some exceptional cases somewhat higher values. Accepting the transported high charge values, e.g. of the order of 100 coulombs, as generally valid also in lightning discharges under open field conditions one is led to field force values within the charge delivering thunderstorm cloud of quite another order of magnitude than the

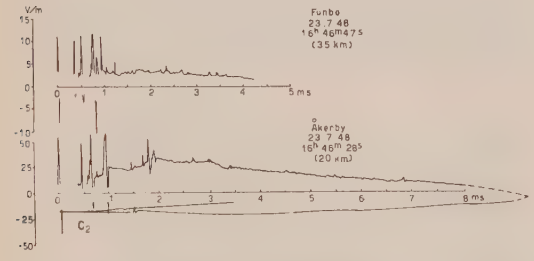


Fig. 5. Two separate lightning discharges recorded at short time intervals at two field stations.

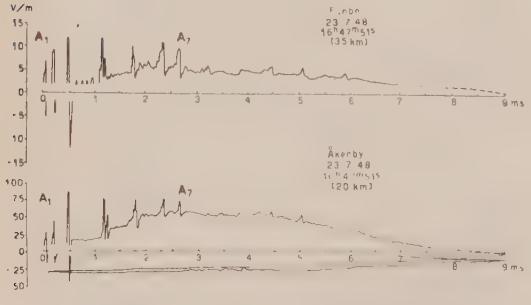


Fig. 6. Simultaneously recorded lightning discharge at two stations. After main discharge A_1 six partial discharges up to A_7. Long after-variation at one station, at the other station it is hidden by zero band arrangement.

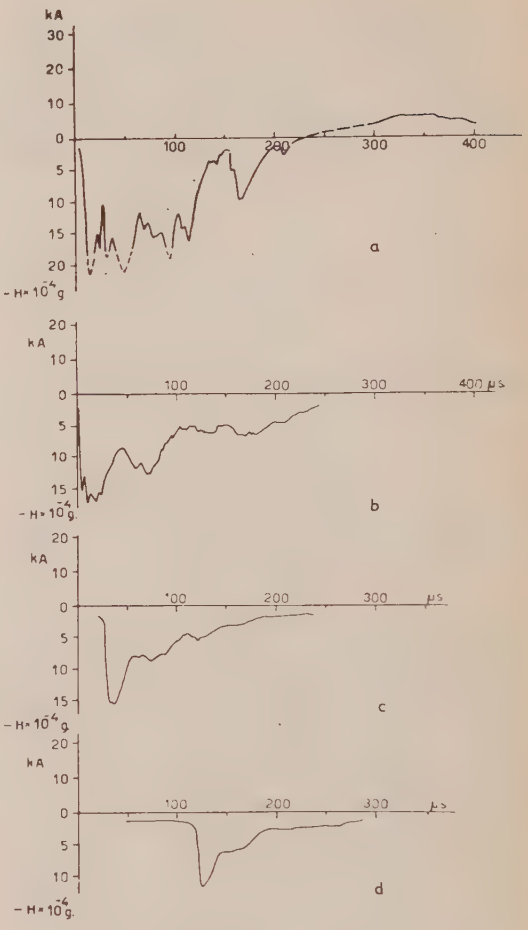


Fig. 7. Current variation diagrams of concecutive lightning strokes in same path. Typical diminuation of amplitudes from first occurring discharge. After ionization effects at c and especially d.

one generally accepted. This discrepancy is not eliminated by the absurd and against all experience contradictory explanation that the surface of the thunderstorm cloud which should be necessary to maintain a lightning discharge, must be 100 square kilometres.

The above mentioned tendency of the maximal current values of successive strokes in the same channel to decrease in most cases can easily be explained. The successive transportations of charges in the multiple strokes must diminish the available charge within the regions of the cloud which are active in the discharges. We have another way to check this typical decreasing tendency of the current intensity in lightning discharges. An extensive investigation of the variation of the magnetic field as caused by vertical lightning discharges has been carried out in Sweden (ref. [3]). Under certain conditions these researches have allowed a calculation of the current variation of successive discharges in the same lightning channel. The decreasing amplitude values are illustrated by fig. 7. In the vertical lightning from which the 4 discharges (a—d) have been oscillographed, they are evidently passing consecutively in the same path. Of special interest is that consecutive discharges in the lightning channel very often possess similar variation forms. From an electrophysical point of view this is quite easy to explain. The reason is that the transportation of the charges takes place in the same channel and hence is fed from the same ionized and active region of charges of the thunderstorm clouds. Worthy of note is the retardation of the steep front especially in d, where a weaker current amplitude precedes the rapidly increasing current front. Evidently this must be explained as being caused by a retention of the ionization from the foregoing discharge in the same channel. Very characteristic is the gradual diminuition of the amplitude of the consecutive current impulses and hence a diminuition of the transported charge. This is easily observed in the curves a — e of fig. 8, which represent the consecutive current discharges in another vertical lightning path.

Of special interest in this connection is the distribution of the total durations of main and partial discharges which constitute vertical lightning paths. Such an analysis has been carried out for 106 individual discharges in the above mentioned earlier investigation at the Institute of High Tension Research (ref. [3] and fig. 9). The real duration values are in this case insignificantly longer, depending on the fact that weaker parts of the recorded field force in the oscillographs are to some extent hidden by the zero-band of the instruments (ref. [4]). The most frequently occuring duration values are from 100 to 200 μsec. This is in full accordance with duration values that have been measured by quite another method (see fig. 2—6). In this case atmospherics were taken up by open antenna circuits. The duration of the rapid main and partial discharges was of the same order of magnitude in fig. 9. Duration values of up to 1000 μsec. are, as may be seen from fig. 9, very rare. It is of importance to keep the quoted duration values in mind when we have to deal with the problem of the total duration of lightning strokes.

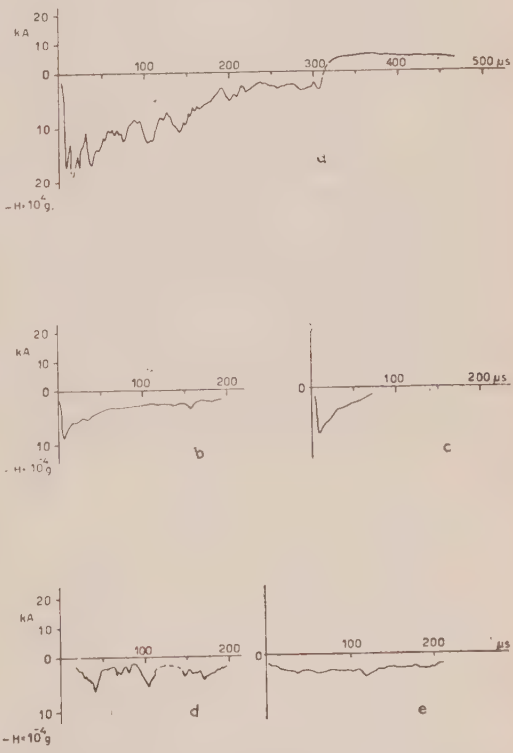


Fig. 8. Current variation diagrams of concecutive lightning strokes in same path. Typical diminuation of current amplitudes from first occurring discharge.

Investigations of Radiated Electromagnetic Field of Lightning as Necessary for Study of Variability of Discharges

The very pronounced variableness, which appears to be characteristic of lightning discharges, requires special precautions if we are trying to carry out a reliable statistical investigation. This must be considered with regard as well to the duration of the discharges, as to the intervals between them and the total duration of lightning strokes. Evidently the primary condition must be to analyse by cathode ray oscillographic recording methods a sufficiently great number of individual lightning strokes. As we shall observe further on it is rather an easy task to obtain extended observations, especially if the researches are extended to include lightning discharges between clouds, and thus not only limited to vertical ones against the earth. In this connection we are able to use extensive experiences from another recent investigation not yet published. This one deals with thousands of oscillographic records of atmospherics taken during the thunderstorm season of 1949. The atmospherics were recorded simultaneously in Sweden with the aid of two stations at a distance of 576 kilometres apart. From this extended accumulation of recorded lightning discharges we are able to conclude that there is no pronounced difference in character, either with regard to duration of individua consecutive discharges in the lightning strokes, or with respect to time intervals between the discharges.

Some examples of typical consecutive lightning discharges recorded from separate lightning paths at one of the two mentioned stations at a distance of 576 km apart, are reproduced in fig. 10—11. In these cases the lightning discharges were located at a considerable distance from the recording station. This must to some extent have influenced the records by reflection effects from ionospheric layers. Such special influences will no tbe discussed in this article. From fig. 10—11, follows that consecutive discharges in the paths of the examples chosen do not show any appreciable differences in their variation character.

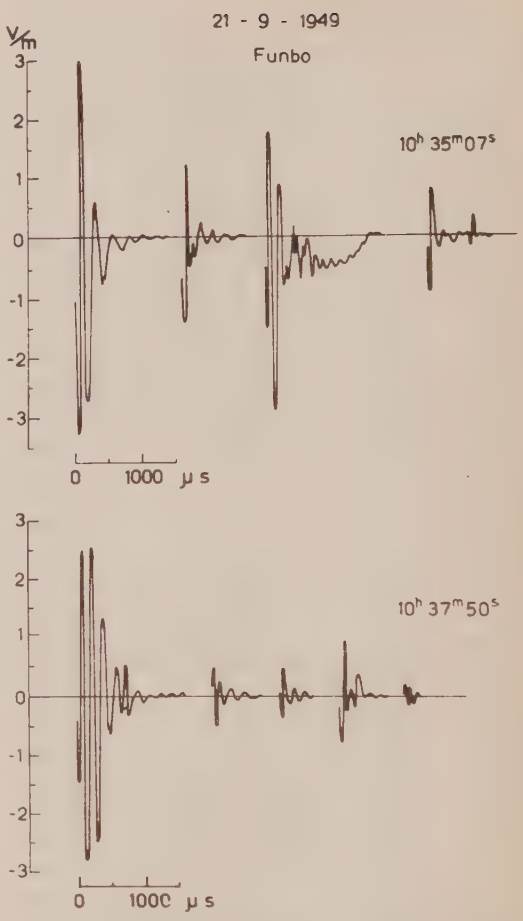


Fig. 10. Consecutive discharges in distant lightning channels recorded at the Institute's field station.

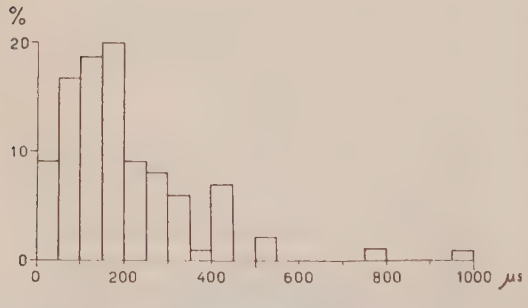


Fig. 9. Duration of currents of individual discharges of 106 vertical lightning discharges recorded at the Institute.

We therefore decided to carry out, during the latter part of the recent thunderstorm season of 1949 in Sweden, an extended analysis of the number of strokes, their total duration and the time interval between multiple discharges in the lightning strokes. For this pur-

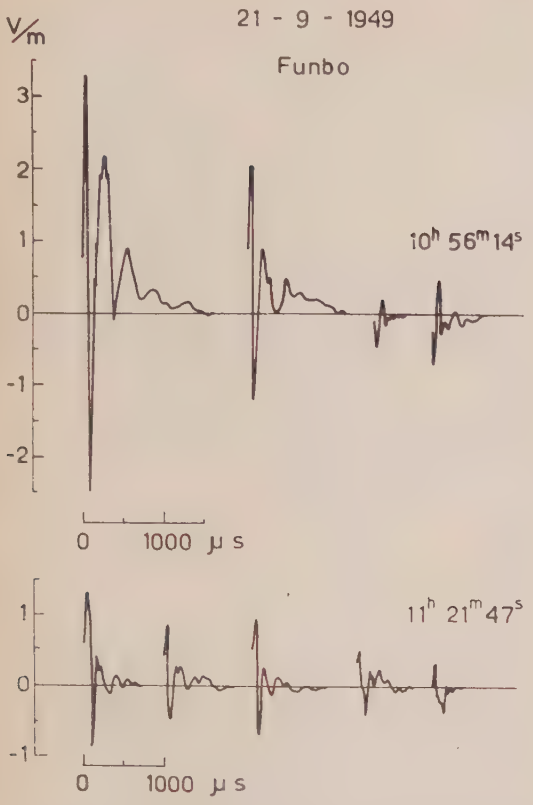


Fig. 11. Consecutive discharges in distant lightning channels recorded at the Institute's field station.

pose we chose a certain day, Oct. 20th 1949, when we had reason to expect only two regions of lightning storms in Sweden. They were situated at such great distances apart that the measurements did not become affected by local disturbances in the vicinity of the thunderstorm centres. The map of fig. 12 illustrates these centres where the filled circles represent damages by strokes to earth-connected structures, such as buildings, telephone lines or electric lines.

In order to measure the lightning discharges from the centres mentioned, an open antenna circuit in combination with a cathode ray oscillograph was used [2]. The method allowed records of the rapid variations of the vertical field force as produced by the lightning strokes from the thunderstorm centres. If the time-base circuit of the oscillograph was adjusted to a suitable time scale we could thus record that each rapid discharge in a lightning stroke — thus the consecutive multiple strokes — appeared as rapid vertical deviations from the zero line of the oscillograph. These vertical deviations were followed by traces of a damped after-variation with a gradually diminishing amplitude. The direction of decrease of this after-variation enabled us to determine the number of multiple strokes and the sequence of them. The records are of a similar type as the ones reproduced in fig. 10—11. From the above mentioned investigation of records of lightning discharges, simultaneously made at two stations far apart, followed that all lightning discharges of appreciable intensity passing in the channel of a lightning stroke, can be recorded provided the amplification of the oscillograph is increased to a sufficiently high level.

Hence, the described method should result in an extended analysis of the number of discharges in lightning strokes. It was also

Fig. 12. Map of distant lightning discharges. Filled circles indicate damages of strokes to ground.

possible to analyse the time intervals between the multiple strokes and the total duration of lightning discharges. The total amount of separate lightning paths, recorded in the investigation, was 257. Of these only 57 consisted of one discharge in the path, 200 carried two or several discharges of which it was possible to measure 686. With the aid of these it was possible to determine 629 intervals.

The collected and analysed records allowed a calculation of the percentage number of multiple strokes reproduced in fig. 13. Evidently, the number carrying only one lightning discharge in the channel must in such a distribution become somewhat preponderant. In this connection it will be of special interest to compare the distribution with the number of strokes in 138 vertical lightning paths see

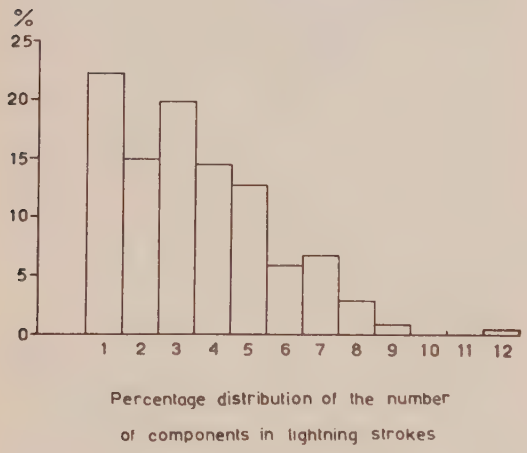


Fig. 13. Percentage distribution of number of components recorded from 257 separate lightning paths.

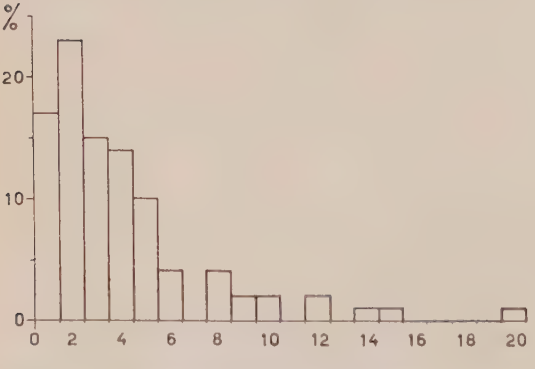


Fig. 14. Percentage distribution of number of components of 138 vertical lightning paths.

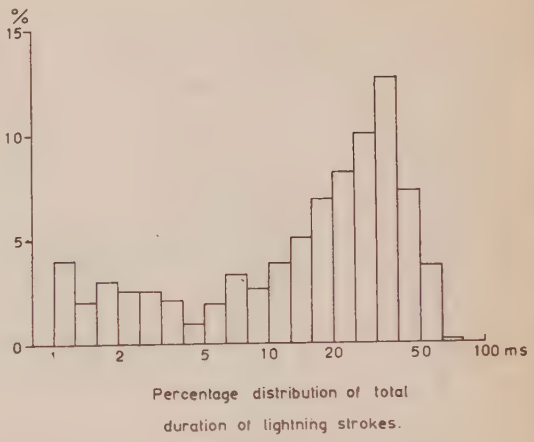


Fig. 15. Percentage distribution of total duration of 250 lightning strokes.

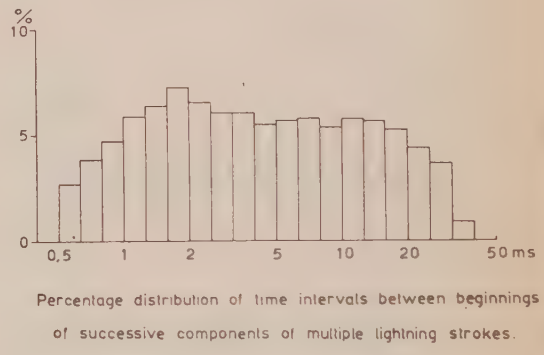
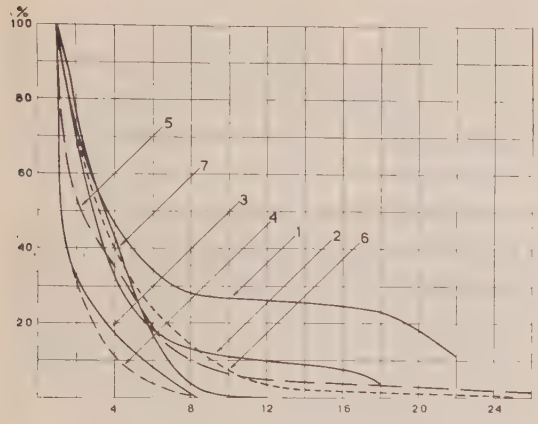


Fig. 16. Distribution of time intervals of successive components of multiple lightning strokes.

fig. 14. This distribution has been taken from an earlier investigation (ref. [3]). When we are considering the shifting variation features of lightning discharges in total we must accept that the distributions in the two cases show a surprisingly good conformity.

Multiple strokes in excess of four are less frequent than the range on four. This is easily accounted for, the more so as the charges of a thunderstorm cloud must be distinguished by a certain limitation. The active charges within the cloud transported by a lightning, can in some cases be diminished to an insignificent value by merely one or a few discharges. From this point of view it will indeed be of a special interest to compare the number of discharges in different types of thunderstorms. Especially a comparison of the conditions within thunderstorms in the temperate zones



Percentage distribution curves of the number of components in lightning strokes

1	Wagner, Mc Cann & Beck	9 strokes to tall objects
2	" " "	37 photographs of strokes to open ground
3	" " "	40 arrester surge currents
4	Mc Eachron	184 protector tube surges
5	Schonland	65 photographs of strokes to open ground
6	Stekolnikov & Valeev	25 induced surges
7	Norinder	257 strokes

Fig. 17. Distribution of number of components in lightning strokes.

with those in the subtropical zones should be of great interest.

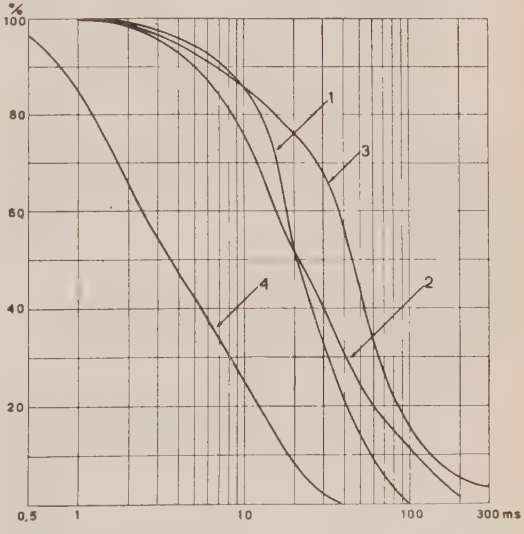
The percentage distribution of the total duration of the multiple discharges, carried out in a logarithmic scale, are reproduced in fig. 15. The most frequently occuring values are varying between 20—40 msec. A percentage comparison, of the durations of the intervals between multiple strokes in the same channel is reproduced in fig. 16. Evidently the duration of the intervals is evenly distributed on a logarithmic cale.

Comparison with Results concerning Duration and Variability of Lightning Discharges in Low Latitude Countries

It is of special interest to compare the results obtained by the investigation of the above mentioned 257 lightning strokes with values obtained by other methods and under different climatic conditions. In this respect it is simple to try a comparison with the data compiled by WAGNER and MC CANN [1]. There are several reasons for such a comparison. The compiled values have been gathered from observations from different countries in the world, all of them situated at much lower latitudes than Sweden.

In order to carry out such a comparison the curves in the paper by WAGNER and MC CANN dealing with the number of multiple strokes, the percentage distribution of time intervals between strokes and the total durations have been redrawn and the recent swedish results have been entered in the redrawn curves. In the original curve systems the authors included some measurements from Sweden taken from my first observations of magnetic field variations from lightning strokes. In this case a differential method was used, which did not allow of a record of the slow moving after-variations of the current pulses. This outcome is quite contrary to the integral methods introduced in a later publication ref. [3]. The quoted older data do not quite correctly reproduce the time intervals between subsequent current pulses and hence they have been excluded from the new curve system of time intervals.

In fig. 17 the percentage number of components in lightning strokes is reproduced. The number of strokes from the Swedish material does not result in a distribution curve with such high values as are valid for strokes to tall objects. The curve obtained from our great number of observations represents a typical average picture and an extension of the obser-



Percentage distribution curves of time intervals between the beginnings of successive components of multiple lightning strokes

1	Wagner, Mc Cann & Beck	strokes to tall objects	35 values
2	" " "	photographs of strokes to open ground	126 "
3	Schonland		109 "
4	Norinder	257 strokes	829 "

Fig. 18. Time intervals between successive components of multiple strokes.

vations will probably not alter this general character.

In fig. 18 is reproduced the percentage distribution curves of time intervals between the beginning of successive components. In this distribution the Swedish values [4] are of a pronounced shorter time variation type, and mostly approach to the curves 1 and 2. The values of 3, taken in South Africa, have pronounced time intervals of longer durations.

In fig. 19 the percentage distribution of durations of lightning strokes is reproduced. Also in this case there is a pronounced difference between the durations as measured in Sweden compared with those in countries situated at lower latitudes.

There is no doubt that the influence of the tall objects upon the results cannot be neglected when the lightning mechanism and especially the time variation character of lightning discharges are considered. The differences in the curves of fig. 19 are pronounced also when we are dealing with measurements where strokes to open ground are analysed. These differences must have another explanation. I am inclined to ascribe the differences to a certain extent to latitude effects, as indicated previously. I have not personally been in a position to investigate lightning discharges in countries situated at a more southerly latitude, but I have had many opportunities to observe thunderstorms visually in southern countries, both in Europe, Africa and North America. My personal impression is that the intensity of such thunderstorms at lower latitudes in most cases is at a much higher level than in Sweden.

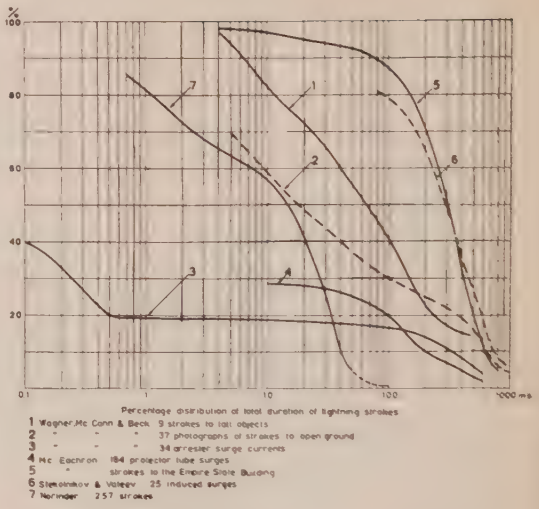

%
100
80
60
40
20
0.1
1
10
100
1000 ms
Percentage distribution of total duration of lightning strokes
1 Wagner, Mc Conn & Beck 9 strokes to tall objects
2 " " " 37 photographs of strokes to open ground
3 " " " 34 arrester surge currents
4 Mc Eachron 184 projector tube surges
5 " " strokes to the Empire State Building
6 Stekolnikov & Valeev 25 induced surges
7 Norinder 257 strokes

Fig. 19. Distribution of total duration of lightning strokes.

Characteristics of Lightning Discharges as Influenced by Latitude Differences of Thunderstorm Intensity

There are several groups of observations which in an evident way manifest that the thermodynamic intensity within thunderstorm clouds very often must be remarkably higher in low latitudes. From this it must follow that the amount of energy transformed into electic charges within the thunderstorms also must attain a higher intensity level. There is no doubt that this quality to some extent must guide to the physical explanation as to why both the lightning current intensity, the length of the intervals between multiple strokes and the total duration of the lightning strokes will sometimes result in marked differences when thunderstorms in higher and lower latitudes are compared. Obviously it should be of special interest to undertake a thorough investigation of this problem. The problem is of importance with regard to the propagation and transformation of atmospherics at long distances. Another aspect must be to what extent lightning currents to the earth are contributing to the maintenance of the negative charge of the earth.

REFERENCES

[1] Wagner, C. F., Mc Cann, G. D., 1944: Lightning Phenomena. Electrical Transmission and Distribution Reference Book. Third Edition p. 291—324. Westinghouse Electric and Manufacturing Company, East Pittsburg.

[2] Norinder, H., 1949: On the Measurement and Nature of Atmospherics Produced by Electric Discharges in Snow Squalls and from other Phenomena. *Tellus* 1, 2.

[3] Norinder, H., Dahle, O., 1946: Measurements by Frame Aerials of Current Variations in Lightning Discharges. *Arkiv för Matematik, Astronomi och Fysik*. 32, A., 8.

[4] Ackermann, O. A., 1937: A Cathode Ray Oscillograph with Norinder Relay. Lightning Reference Book p. 564. Amer. Inst. of Electrical Engineers, New York.

Adiabatic Perturbation Equations for a Zonal Atmospheric Current

By P. QUENEY, University of Paris

(Manuscript received 25 November 1949)

Abstract

By a proper choice of coordinates and variables, a simple and quasi-symmetrical system of differential equations is obtained for the infinitesimal adiabatic perturbations of the most general continuous zonal air-flow. The coordinate surfaces are those of the orthogonal system built up with the isentropic surfaces and the meridians, and the variables are the displacements of the fluid in the directions of the coordinate lines, and the local disturbance of the pressure. From these perturbation equations a differential equation for the pressure disturbance alone is derived in the case of a wave, and it is then shown that this wave equation can be always greatly simplified if account is taken of the order of magnitude of the various parameters. Simplified expressions are also given for the components of the vibration (fluid displacement). More particular equations and formulae are then derived for short-period, middle-period and long-period waves successively, this classification being made according to the value of the orbital frequency compared to the coefficient of hydrostatic stability and Coriolis parameter, and special cases are also considered where the wave equation reduces to one with constant coefficients (elastic, gravitational, Rossby's waves). Finally some incorrect recently published results are discussed.

1. Definition and meteorological significance of the perturbation problem

It is well known that when a fluid motion may be considered as a *disturbed motion*, that is, resulting of an infinitesimal perturbation superimposed upon a given *undisturbed motion*, its differential equations can be linearized, in other words replaced with a system of linear equations, the so-called *perturbation equations*, the resolution of which is of course easier than that of the motion equations themselves, at least when the undisturbed motion is a simple one.

However, when considered from this viewpoint, the perturbation problem (resolution of the perturbation equations under given conditions) might look to be one of only mathematical interest, and the large place which it takes in modern meteorology would not be justified. It seems therefore necessary to give here a brief discussion about this topic: what is the practical significance of the perturbation problem in meteorology?

During the past two centuries, the perturbation method was used by several of the greatest mathematicians or physicists when they built the now classical theories of such phenomena as sound-waves (NEWTON, LAPLACE, POISSON, STOKES, RAYLEIGH), ocean's tides (LAPLACE), gravitational water-waves (LAGRANGE, CAUCHY, POISSON, STOKES, GREEN, AIRY, KELVIN), capillary-gravitational waves (KELVIN). Nobody can deny the great success of these theories for the explanation of important observed facts, but it must also be noted that most of them only concern very simple problems: the fluid in the undisturbed state is stagnant or in uniform motion, and possesses no internal stratification, so that the coefficients of the perturbation equations are generally constant, and the perturbations are essentially stable.

The first example of unstable waves was given by HELMHOLTZ 1868 when he considered the oscillations of the system formed by two homogeneous currents moving side by side with uniform but different velocities (shearing-gravitational waves). In this case, the shear at the common boundary is the only factor of instability, and the only stabilizing factor is gravity, which operates mainly on long waves. From this it follows that all the waves are unstable the wave-length of which is smaller than a critical value, whence HELMHOLTZ concluded that a turbulent layer of transition must develop between the two currents, the width of this layer being comparable to the critical wave-length (in modern terminology, the system is said to be "dynamically unstable", or to possess a "dynamic instability"). HELMHOLTZ applied his theory to stationary billow-clouds, by assuming that the surface of discontinuity is the upper boundary of an atmospheric inversion.

In the case of air over water, this problem was generalized by KELVIN 1871, by introducing capillarity as an additional stabilizing factor, and this author made an application to the theory of the formation of water-waves by wind. Since capillarity mainly operates on short waves, and gravity on long waves, it is found that the system is dynamically unstable only if the wind exceeds some critical value, and if the wind increases from this value, there is a limited spectral-interval of unstable waves, the width of which increases from zero. We have here a simple example of an "instability criterium" (condition for the undisturbed motion to be dynamically unstable), which may be physically interpreted as the condition for the motion to change from laminar to turbulent type in the vicinity of the surface of water.

If the theory be correct, the critical velocity would be that of the slightest wind which is able to raise waves on a smooth water-surface, and the theory would also give the wave-length of these first-appearing waves. As a matter of fact, it is actually observed that there is a definite wind-velocity at which the surface of water begins to be ruffled, but this velocity is much smaller than KELVIN's critical value (about six times smaller). The explanation of this disagreement, as pointed out by JEFFREYS 1925, is probably that a turbulent air-layer begins to form on the lee-side of each wave at a wind-velocity which is much smaller than the critical value, thus increasing the effect of wind-shear as a factor of instability. But there is still no complete theory of wind-waves.

It has been noticed for a long time that many of the atmospheric perturbations have the same character as gravitational or shearing-gravitational waves, but this analogy remained a purely formal one as long as the general structure of the atmosphere was not known.

The earliest theory concerning atmospheric waves was perhaps HELMHOLTZ's already mentioned solution of the billow-clouds problem, and a quite similar theory is that proposed by KÜTTNER 1939 for mountain lee-waves (in both cases an inversion boundary is assumed to exist).

But by far the most interesting surface-wave theory applicable to the atmosphere appeared after J. BJERKNES and his co-workers discovered the atmospheric fronts in 1919, and were able to describe the cyclones of Northern Europe as wavy disturbances growing and progressing along a quasi-permanent temperature-discontinuity, the so-called polar-front surface. This interpretation could not fail to suggest a close analogy between cyclone-waves and wind-waves for instance, and therefore it was natural to consider the theoretical problem of the oscillations of the system formed by two air currents having uniform but different temperatures and velocities, separated by the polar-front surface and bounded downwards by a horizontal surface at sea level. Unfortunately this problem is much more difficult than KELVIN's wind-wave problem, specially because of the slope of the polar-front surface, which cannot be neglected at the scale of cyclone-waves. Tentative approximate solutions were given by SOLBERG 1928 and then by V. BJERKNES and SOLBERG 1933, and the general results were quite similar to KELVIN's ones concerning wind-waves: owing to the fact that there are two stabilizing factors, namely the gravity, operating now mainly on the shortest waves (the very short waves, of the scale of billow-clouds, are disregarded in this problem), and the earth's rotation, which on the contrary mainly operates on the longest waves, the system is dynamically stable as

long as the wind shear is smaller than a critical value, and when this value is just exceeded a dominant unstable wave appears, which may be interpreted as the cyclone wave.

This remarkable result, following the great push of enthusiasm created by J. BJERKNES' discovery, largely contributed to the success of the Norwegian meteorological school. However, when considered objectively, this result is in reality more a disapproval than a justification of J. BJERKNES' fundamental idea that cyclones are the result of the conflict between two air masses of different temperatures: in V. BJERKNES-SOLBERG's waves, a temperature difference is indeed a stabilizing factor, the only factor of instability being the wind shear. On the other hand, it very soon appeared that the polar-front surface is not a continuous surface, that it never exists out of cyclones, and that many extra-tropical cyclones, for instance those of the mediterranean area, do not show any front at their early stage. It was therefore obvious, in the years preceding the last world war, that the problem of cyclone formation had to be completely reexamined.

In presence of this failure, the modern meteorologists were forced to abandon the idea that atmospheric perturbations are necessarily related to discontinuities, and they were led to consider, as factors of stability and instability, those arising from the internal stratification and wind-shear of the atmospheric layers. The simplest cases being obviously those where there is no internal wind-shear, one must expect to find here the first successful attempts of this recent stage of theoretical meteorology. Let us mention specially the beautiful results arrived at by LAMB 1932, TAYLOR 1936 and PEKERIS 1937, concerning the atmospheric tides, ROSSBY's well-known theory of long waves in the westerlies, and also, as provisional theories, HAURWITZ's results about billow-clouds, and the recent researches on mountain waves.

But it is clear that, if the assumption of no shear is maintained, there can be no hope to arrive at an even approximate theory for atmospheric perturbations of the scale of cyclones, since these perturbations are always associated with important horizontal temperature-gradients, and these in their turn are associated with vertical wind-shears. This consideration explains why the cyclone problem has become practically identified with the problem of unstable waves in a baroclinic atmospheric current, and why so much importance is attached in present days to the derivation of instability criteria for such currents. Unfortunately, the results obtained up to now in this connection are rather discordant: according to a first group of meteorologists (SOLBERG 1936, HOEILAND 1939 and 1941, FJOERTOFT 1941 and 1946, VAN MIEGHEM 1944 and 1946), it is possible to derive, at every point of a baroclinic air flow, an instability criterium which only involves the *local* values of hydrodynamical parameters, while according to others (JAW 1946, CHARNEY 1947, EADY 1949), on the contrary, the *boundary conditions* play a dominant rôle, and for the current values of the wind-shear all the short waves are unstable as far as the vertical acceleration is neglected (the very short waves are stable, but they are unimportant in the problem), in other words the system is practically always dynamically-unstable. (In order to reconcile this result with the existence of a dominant wave-length in the turbulent motion, Eady imagines to introduce the postulate that, among the numerous unstable waves, it is the wave of maximum growth-rate which must predominate.)

What may be now the conclusion of this brief historical survey? The failure of the perturbation method when applied to shearing or baroclinic systems is probably to be attributed, not to the method itself, but well to the incorrectness of the assumptions made by the authors of theories, concerning the hydrodynamical conditions imposed to the motion: in the wind-wave problem, for instance, the motion must be taken as turbulent instead of laminar as in TAYLOR's theory, and in all of the recent researches about baroclinic currents, the simplifying assumptions made in course of the computations are generally not permissible, as we shall prove it for some of them at the end of this paper. It is therefore to be expected that, once the perturbation problem is expressed in correct analytical form, and the mathematical derivation are correctly made, the same success will be obtained for baroclinic as for stagnant fluids.

However it is obvious that the perturbation method will be always quite unable to explain and forecast the growth of perturbations inside

the turbulent regions which result from dynamic instability. This is in fact a very serious limitation of the meteorological significance of the perturbation problem, and it may be that most of cyclonic perturbations, for instance, have nothing to do with dynamic instability (this is strongly suggested by J. BJERKNES' original theory). However there is certainly a great deal of meteorological features such as long waves, mountain waves, maintainance of turbulence (cf. ROSSBY 1947), etc., which will be largely explained by using perturbation method.

2. General scope of the present paper

In the preceding section we have mentioned that, because of the great difficulties encountered in the mathematical derivations, most of the authors of recent researches about the perturbations of baroclinic currents have been led to introduce several simplifying assumptions in the course of the computations, and that these assumptions are generally not permissible. Since the most interesting problems are probably those related to zonal currents, we have found it important to give a system of perturbation equations for the most general continuous zonal stream, and to show that these equations can be practically always written in a relatively simple form if proper variables and coordinates are used, when the parameters are taken with the usual order of magnitude observed in the lower (or meteorological) atmosphere (up to 30 km, say). This is the main scope of this paper. In the last sections, we also give some applications to simple cases, and a brief discussion about some incorrect recently published results.

3. Definition and general properties of symbols

a) A_j, abridged notation for $\partial A/\partial j$

D/Dt, individual derivative with respect to time t

$\overline{A}$, value of A in undisturbed motion

∂A, local disturbance of A, that is, variation of A, from undisturbed to disturbed motion, at fixed time and point

DA, individual disturbance of A, that is, variation of A at fixed time and fluid-element

(the computation rules for differentials hold for the operators ∂ and D separately)

(A), order of magnitude of A

$A \sim B$ (or $B \sim A$); $A \gg B$ (or $B \ll A$); $A \gtrsim B$ (or $B \gtrsim A$), means that $(A) = (B)$; $(A) > (B)$; $(A) \geqslant (B)$, respectively

$A \# B$ (or $B \# A$), means that $A - B \ll A$

b) $\mathbf{M}$, position vector of the current point M defined by the space coordinates q_1, q_2, q_3

($D\mathbf{M}$ is the displacement of the fluid-element from undisturbed to disturbed motion)

Dx_1, Dx_2, Dx_3, components of $D\mathbf{M}$ along x_1, x_2, x_3, directions of the three coordinate lines

$\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$, partial derivatives in the directions x_1, x_2, x_3

$$D\overline{A} = A_{q1}\, Dq_1 + A_{q2}\, Dq_2 + A_{q3}\, Dq_3 = \\ = A_{x1}\, Dx_1 + A_{x2}\, Dx_2 + A_{x3}\, Dx_3 \qquad (3.1)$$

$$DA = \partial A + D\overline{A} \qquad (3.2)$$

$$\partial\,(\partial A/\partial j) = (\partial/\partial j)\,(\partial A);\ D(DA/Dt) = \\ = (D/Dt)\,(DA) \qquad (3.3)$$

c) i, imaginary unit; $\log A$, Neperian logarithm of A; $\mathbf{A} \cdot \mathbf{B}$, salar product; $\mathbf{A} \times \mathbf{B}$, vectorial product

d) L, r, h, cylindrical absolute coordinates relative to earth's axis: L, absolute longitude; r, distance to earth's axis; h, distance above equatorial plane

x, y, z, axes pointing eastwards, northwards, upwards, respectively

x, Y, Z, direct system of rectangular axes such that Y is directed northwards parallel to isentropic surface

$\mathbf{x}$, unit vector of x-direction

e) g, relative gravity (that is, ordinary gravity); E, potential of absolute gravity; p, pressure; q, density; T, absolute temperature; Θ, potential temperature; $m = c_v/c_p$, ratio of specific heats;

$a = m\, q/p$, coefficient of piezotropy for adiabatic changes

$s = \log\,(p^m/q) = \log \Theta + \text{const}$, a linear function of specific entropy

$s' = \log\,[(1/p^m)\sqrt{q/r}] = \frac{1}{2}\,[\log\,(1/rp^m) - s]$; $s'' = s' + \log\sqrt{s_z}$

$N_0 = \sqrt{gs_z}$, coefficient of hydrostatic stability

f) k; N; $P = 2\pi/N$, wave-number, orbital angular-frequency and orbital period of a sine wave in x-direction, respectively

g) $D'x$, $D'y$, $D'z$, $D'Y$, $D'Z = \sqrt{qr}\,(Dx, Dy, Dz, DY, DZ)$

$\partial' p = \sqrt{r/q}\,\partial p$; $\partial'' p = \partial' p/\sqrt{s_z}$

h) φ, latitude; $\tan\varepsilon = s_y/s_z$, slope of the isentropic surface

$\Phi = \varphi + \varepsilon$, angle of Y with equatorial plane

$1/R = \varphi_y$, meridian curvature of the level surface

ω, absolute angular velocity about earth's axis

$\mathcal{W}$, vector of magnitude ω parallel to earth's axis

ω_0, earth's angular velocity

$u = r\,(\omega - \omega_0)$, eastward velocity relative to earth

$f = 2\,\omega_0 \sin\varphi$, Coriolis parameter

$F = 2\,\omega \sin\varphi$, vertical component of $2\,\mathcal{W}$

$F_1 = 2\,\omega \sin\Phi$; $F_2 = 2\,\omega\cos\Phi$, components of $2\,\mathcal{W}$ along Z and Y, respectively

$N_1^2 = -\,(p_Z/q)\,s_Z + F_2 r\,\omega_Z$

$\varepsilon' = F_2^2/(N_1^2 - N^2)$; $F' = F_1 + \varepsilon' r\,\omega_Y$

$F_1' = F_1 - r\,\omega_Y$, component of absolute vorticity normal to the isentropic surface

$G_1 = F_1 F_1' - N^2$; $G' = G_1 + \varepsilon'\,(F_1' r\,\omega_Y - N^2)$

$G = F\,(F - u_Y) - N^2$

$B = (1/G)\,[F^2\,(F - u_Y)_Y + F_Y N^2] + (F/s_z)\,(aFu_z + s_{zY})$

4. Specification of the motions

The undisturbed motion is assumed to be a zonal flow of *limpid* and *unviscid* air, defined by *continuous* meridian distributions of ω and of two of the parameters p, q, T, Θ, a, s defined in (3.e), these three distributions being related together by the motion equations, given in next section.

The perturbation is simply assumed to be adiabatic and laminar.

5. Vectorial perturbation equations

If use is made of the cylindrical coordinates L, r, h defined in (3.d), the mechanical equations for an unviscid fluid motion may be written as follows:

$$\left.\begin{aligned} &\frac{D}{Dt}\left(r^2\frac{DL}{Dt}\right) + \frac{p_L}{q} = 0 \\ &\frac{D^2 r}{Dt^2} - r\left(\frac{DL}{Dt}\right)^2 + \frac{p_r}{q} + E_r = 0 \\ &\frac{D^2 h}{Dt^2} + \frac{p_h}{q} + E_h = 0 \end{aligned}\right\} \quad (5.1)$$

Applied to the undisturbed motion, they give

$$p_L = 0;\ (p_r/q) + E_r = r\omega^2;\ (p_h/q) + E_h = 0$$

If we differentiate the last two of these relations with respect to h and r respectively, and then subtract, we get

$$(1/q^2)\,(q_r p_h - q_h p_r) = 2\,\omega\, r\,\omega_h \qquad (5.2)$$

or, introducing the parameter s defined in (3.e),

$$(1/q)\,(s_h p_r - s_r p_h) = 2\,\omega\, r\,\omega_h \qquad (5.3)$$

and these relations are two equivalent forms of the so-called geostrophic equilibrium condition. It follows from (5.2) that the undisturbed motion is barotropic if, and only if, ω is independent of h.

Instead of (5.3) we may write, in vectorial form,

$$\nabla s \times (\nabla p/q) = (2\,\mathcal{W}\cdot r\nabla\omega)\,\mathbf{x} \qquad (5.4)$$

The mechanical perturbation equations are readily deduced from (5.1) by applying the operator ∂ to the three equations of this system. We thus get, using (3.3),

$$\left.\begin{aligned} &\partial\frac{D}{Dt}\left(r^2\frac{DL}{Dt}\right) + \frac{1}{q}\frac{\partial}{\partial L}\partial p = 0 \\ &\partial\frac{D^2 r}{Dt^2} - r\,\partial\left(\frac{DL}{Dt}\right)^2 + \frac{1}{q}\frac{\partial}{\partial r}\partial p - p_r\frac{\partial q}{q^2} = 0 \\ &\partial\frac{D^2 h}{Dt^2} + \frac{1}{q}\frac{\partial}{\partial h}\partial p - p_h\frac{\partial q}{q^2} = 0 \end{aligned}\right\}$$

But in the undisturbed motion we have

$$\frac{D}{Dt}\left(r^2\frac{DL}{Dt}\right) = 0;\ \frac{D^2 r}{Dt^2} = 0;\ \frac{D^2 h}{Dt^2} = 0;\ \frac{DL}{Dt} = \omega$$

whence, using (3.2) and (3.3), and since $rDL = Dx$,

$$\partial \frac{D}{Dt}\left(r^2 \frac{DL}{Dt}\right) = D \frac{D}{Dt}\left(r^2 \frac{DL}{Dt}\right) = r^2 D \frac{D^2 L}{Dt^2} +$$

$$+ \omega D \frac{Dr^2}{Dt} = r^2 \frac{D^2}{Dt^2} DL + 2\omega r \frac{D}{Dt} Dr =$$

$$= r\left(\frac{D^2}{Dt^2} Dx + 2\omega \frac{D}{Dt} Dr\right);$$

$$\partial \frac{D^2 r}{Dt^2} = D \frac{D^2 r}{Dt^2} = \frac{D^2}{Dt^2} Dr; \quad \partial \frac{D^2 h}{Dt^2} = \frac{D^2}{Dt^2} Dh;$$

$$r\partial \left(\frac{DL}{Dt}\right)^2 = 2\omega r \, \partial \frac{DL}{Dt} -$$

$$= 2\omega r\left(D \frac{DL}{Dt} - D\bar{\omega}\right) = 2\omega\left(\frac{D}{Dt} Dx - rD\bar{\omega}\right).$$

We thus arrive at the system

$$\left.\begin{aligned} &\frac{D^2}{Dt^2} Dx + 2\omega \frac{D}{Dt} Dr + \frac{1}{q} \frac{\partial}{\partial x} \partial p = 0 \\ &\frac{D^2}{Dt^2} Dr - 2\omega\left(\frac{D}{Dt} Dx - r D\bar{\omega}\right) + \frac{1}{q} \frac{\partial}{\partial r} \partial p - \\ &\qquad - \frac{p_r}{q} \frac{\partial q}{q} = 0 \\ &\frac{D^2}{Dt^2} Dh + \frac{1}{q} \frac{\partial}{\partial h} \partial p - \frac{p_h}{q} \frac{\partial q}{q} = 0 \end{aligned}\right\}$$

which is equivalent to the vectorial equation

$$\frac{D^2}{Dt^2} D\mathbf{M} + 2\omega \times \left[\frac{D}{Dt} D\mathbf{M} - (rD\bar{\omega})\,\mathbf{x}\right] +$$

$$+ \frac{1}{q} \nabla \partial p - \frac{\nabla p}{q} \frac{\partial q}{q} = 0 \qquad (5.5)$$

Let us now consider the continuity equation and the thermal equation. For the motion itself they may be written respectively, if the air is treated as a perfect gas,

$$\operatorname{div} \frac{D\mathbf{M}}{Dt} + \frac{1}{q} \frac{Dq}{Dt} = 0; \quad \frac{Ds}{Dt} = 0$$

and the corresponding perturbation equations may be written at once from analogy, if a proper choice is made of the correspondence between the fluid elements in the undisturbed and in the disturbed motion, as follows:

$$\operatorname{div} D\mathbf{M} + \frac{Dq}{q} = 0; \; Ds = 0 \qquad (5.6)$$

From the last one of these equations we get, according to the definition of s,

$$\frac{\partial q}{q} = m \frac{\partial p}{p} + D\bar{s}; \quad \frac{Dq}{q} = \frac{a}{q} \partial p + m \frac{D\bar{p}}{p}$$

and if we substitute in (5.5) and the first equation (5.6), we get the desired system of vectorial perturbation equations as follows:

$$\left.\begin{aligned} &\frac{D^2}{Dt^2} D\mathbf{M} + 2\omega \times \left[\frac{D}{Dt} D\mathbf{M} - (rD\bar{\omega})\,\mathbf{x}\right] - \\ &- \frac{\nabla p}{q} D\bar{s} + \frac{1}{q}\left(\nabla - m \frac{\nabla p}{p}\right) \partial p = 0 \end{aligned}\right\} \qquad (5.7.1)$$

$$\left.\begin{aligned} \operatorname{div} D\mathbf{M} + m \frac{D\bar{p}}{p} + \frac{a}{q} \partial p = 0 \end{aligned}\right\} \qquad (5.7.2)$$

6. Perturbation equations referred to isentropic coordinates

Once a coordinate system is chosen, it is easy to derive from (5.7) a set of scalar equations in which the unknowns are ∂p and Dx_1, Dx_2, Dx_3, projections of $D\mathbf{M}$ on the directions of the coordinate lines: these equations are obtained by substituting to (5.7.1) its projections on these three directions, expressing $D\bar{\omega}$, $D\bar{s}$ and $D\bar{p}$ in terms of Dx_1, Dx_2 and Dx_3 by means of (3.1), and likewise for div $D\mathbf{M}$ by means of Gauss' divergence theorem for instance. We thus get a system of four differential equations which are obviously linear and homogeneous with respect to the four unknowns, and therefore form a complete system of perturbation equations.

In the present case of an adiabatic motion, it is found that the simplest equations are obtained if reference is made, not to the usual coordinate lines formed by parallels, meridian level-lines and verticals, i.e. tangent at each point to the x, y, z-axes, but to the similar system tangent to the x, Y, Z-axes (see (3.d) for definitions). A coordinate system corresponding to such coordinate lines may be conveniently called an *isentropic system*.

If isentropic coordinates are used, we have simply $D\bar{s} = s_Z DZ$ since $s_Y = 0$, and the system of perturbation equations equivalent to (5.7) may be written as follows:

$$\left.\begin{aligned}
&\frac{D^2}{Dt^2} Dx - F_1 \frac{D}{Dt} DY + F_2 \frac{D}{Dt} DZ + \\
&\qquad + \frac{1}{q} \frac{\partial}{\partial x} \partial p = 0, \\
&F_1 \frac{D}{Dt} Dx + \left(\frac{D^2}{Dt^2} - F_1 r \omega_Y\right) DY + \\
&\quad + \left(-\frac{p_Y}{q} s_Z - F_1 r \omega_Z\right) DZ + \\
&\quad + \frac{1}{q} \left(\frac{\partial}{\partial Y} - m \frac{p_Y}{p}\right) \partial p = 0, \\
&- F_2 \frac{D}{Dt} Dx + F_2 r \omega_Y DY + \\
&+ \left(\frac{D^2}{Dt^2} + N_1^2\right) DZ + \frac{1}{q} \left(\frac{\partial}{\partial Z} - m \frac{p_Z}{p}\right) \partial p = 0, \\
&\frac{\partial}{\partial x} Dx + \left(\frac{\partial}{\partial Y} + m \frac{p_Y}{p} + \frac{r_Y}{r} - \Phi_Z\right) DY + \\
&+ \left(\frac{\partial}{\partial Z} + m \frac{p_Z}{p} + \frac{r_Z}{r} + \Phi_Y\right) DZ + \frac{a}{q} \partial p = 0.
\end{aligned}\right\} \tag{6.1}$$

(see (3.h) for symbols).

The quantities Φ_Z and Φ_Y which appear in the last equation are equal to the curvatures of the coordinate lines tangent to Y and Z respectively, and the following relations can be established:

$$\Phi_Z = \frac{s_{ZY}}{s_Z}; \quad \varphi_Y = \frac{\cos \varepsilon}{R} - \frac{g_y}{g} \sin \varepsilon \tag{6.2}$$
$$(\Phi_Y = \varphi_Y + \varepsilon_Y)$$

On the other hand, the geostrophic equilibrium condition (5.4) takes the form

$$-\frac{p_Y}{q} s_Z = F_1 r \omega_Z + F_2 r \omega_Y \tag{6.3}$$

whence we conclude that the coefficient of DZ in the second equation (6.1) is equal to the coefficient of DY in the third one, and a glance at the other coefficients of the system reveals that most of their terms are either symmetrical or anti-symmetrical with res, ect to the principal diagonal. (The same result holds with any other meridian coordinates.) The symmetry is however incomplete because of the presence of the parameters $1/q$, r_Y/r, r_Z/r, Φ_Z and Φ_Y, but it is easy to get rid of the first three of them, by using as new unknowns the quantities $D'x$, $D'Y$, $D'Z$, $\partial' p$ defined in (3.g). We thus get the following system, equivalent to (6.1) and obviously simpler:

$$\left.\begin{aligned}
&\frac{D^2}{Dt^2} D'x - F_1 \frac{D}{Dt} D'Y + F_2 \frac{D}{Dt} D'Z + \\
&\qquad + \frac{\partial}{\partial x} \partial' p = 0, \\
&F_1 \frac{D}{Dt} D'x + \left(\frac{D^2}{Dt^2} - F_1 r \omega_Y\right) D'Y + \\
&+ F_2 r \omega_Y D'Z + \left(\frac{\partial}{\partial Y} + s'_Y\right) \partial' p = 0, \\
&- F_2 \frac{D}{Dt} D'x + F_2 r \omega_Y D'Y + \\
&+ \left(\frac{D^2}{Dt^2} + N_1^2\right) D'Z + \left(\frac{\partial}{\partial Z} + s'_Z\right) \partial' p = 0, \\
&\frac{\partial}{\partial x} D'x + \left(\frac{\partial}{\partial Y} - s'_Y - \Phi_Z\right) D'Y + \\
&+ \left(\frac{\partial}{\partial Z} - s'_Z + \Phi_Y\right) D'Z + a \partial' p = 0.
\end{aligned}\right\} \tag{6.4}$$

where s' is the parameter defined in (3.e).

It is even possible to go further, and to get rid also of the term Φ_Z by taking as unknowns the quantities

$$D'x/\sqrt{s_Z}; \quad D'Y/\sqrt{s_Z}; \quad \sqrt{s_Z}\, D'Z; \quad \partial' p/\sqrt{s_Z}$$

The symmetry of the corresponding system of equations is practically perfect, since Φ_Y is always small compared to s'_Z. However, in view of the following developments, it is more convenient to retain the system (6.4) itself, which will be therefore our *fundamental system of perturbation equations.*

7. Zonal waves and wave equation

A system of differential equations such as (6.4), the coefficients of which are independent of L and t, is satisfied by solutions of the form

$$(D'x,\ D'Y,\ D'Z,\ \partial'p) = (D'x_0,\ D'Y_0,\ D'Z_0,\ \partial'p_0)\ \exp\ i\,(nL - n't)$$

where n is a real positive integer, n' a real or complex constant and $D'x_0$, $D'Y_0$, $D'Z_0$, $\partial'p_0$ are functions of the meridian coordinates only. The corresponding perturbations may be conveniently referred to as *zonal waves*, for which we may speak of a wave-number k and an orbital angular-frequency N, defined by

$$k = n/r;\ N = n\omega - n' \qquad (7.1)$$

so that both of these parameters are, as a rule, functions of the meridian coordinates.

In order to get the perturbation equations for such waves, we only have to make in (6.4) the substitutions

$$\partial/\partial x = i\,k;\ D/Dt = i\,N \qquad (7.2)$$

If this is done, we observe that in the first three equations no differential operator figures in the coefficients of $D'x$, $D'Y$ and $D'Z$. We can therefore easily solve these three equations for $D'x$, $D'Y$ and $D'Z$, and substituting the results in the last equation we get a differential equation for $\partial'p$ only, which we shall refer to as the *wave equation* of the perturbations.

In the wave equation, the differential operators $\partial/\partial x$ and D/Dt may be reintroduced in place of ik and iN respectively if these parameters have been first removed from the denominator, and the differential equation obtained in this fashion may be more convenient than the wave equation itself to give the solution of a perturbation problem when the boundary conditions are not simple. However in many cases the solution is more readily obtained by using the waves as elementary solutions, according to Fourier's general method.

8. Simplification of the wave equation

The wave equation generally appears with a rather complex form, but it happens that it can always be greatly simplified if account is taken of the order of magnitude of the parameters. It is important to note here that *the simplification must be made only once the wave equation is completely derived*, and not in the course of this derivation. We shall see at the end of the paper to what kind of errors one may be led by the non-observation of this fundamental rule.

We must first observe that, as a rule, the scale of space-distribution of $\partial'p$ and of all its partial derivatives is smaller than, or at most comparable to, that of any parameter Q involved in the wave equation, which we shall refer to as a *fundamental parameter*. Otherwise the portion of atmosphere under consideration would appear as a surface of discontinuity for Q, as far as the perturbation is concerned, and this would not agree with our assumption that the motion is everywhere continuous.

If P denotes the variable $\partial'p$ or any of its partial derivatives with respect to Y or Z, we may thus write, as a rule,

$$Q_Y P \gtrsim Q\frac{\partial}{\partial Y}P;\ Q_Z P \gtrsim Q\frac{\partial}{\partial Z}P \qquad (8.1)$$

whence

$$\frac{\partial}{\partial Y}Q\,P \sim Q\frac{\partial}{\partial Y}P;\ \frac{\partial}{\partial Z}Q\,P \sim Q\frac{\partial}{\partial Z}P \qquad (8.2)$$

In other words, *the order of magnitude of a term of the wave equation is not changed by an arbitrary permutation between the parameters and the differential operators.*

The relations (8.1) and (8.2) still hold, as a rule, if Q is replaced with a function Q' of fundamental parameters, provided that Q' is not simply the difference $Q_1 - Q_0$ between two approximately equal quantities ($Q_1 \# Q_0$), or a function of such a difference. Therefore if a fundamental parameter may be put in the form

$$Q = Q_0 + Q',\ \text{with}\ Q' \ll Q_0\ (\text{or}\ Q \# Q_0)$$

we can write

$$\frac{\partial}{\partial Y}Q'\,P \sim Q'\frac{\partial}{\partial Y}P \lesssim Q_0\frac{\partial}{\partial Y}P \sim \frac{\partial}{\partial Y}Q_0\,P$$

and likewise with the operator $\partial/\partial Z$. We thus get

$$\frac{\partial}{\partial Y} Q P \# \frac{\partial}{\partial Y} Q_0 P;\ \frac{\partial}{\partial Z} Q P \# \frac{\partial}{\partial Z} Q_0 P \quad (8.3)$$

a formula which is very convenient to simplify the coefficients of the wave equation.

Considering now two arbitrary fundamental parameters Q_1 and Q_2, we can easily verify, by assuming successively $Q_1 \gtrsim Q_2$ and $Q_1 \lesssim Q_2$, that, as a rule,

$$Q_1 + Q_2 \sim Q_1 - Q_2 \quad (8.4)$$

$$Q_1 Q_2 \gtrsim Q_1^2 \pm Q_2^2 \sim (Q_1 \pm Q_2)^2 \quad (8.5)$$

and these relations still hold if Q_1 and Q_2 are replaced with arbitrary symbolic functions of fundamental parameters and differential operators applied to the variable $\partial' p$, so that we have, for instance,

$$\left(\frac{\partial}{\partial Y} + Q_1\right) P \sim \left(\frac{\partial}{\partial Y} - Q_1\right) P,$$

$$Q_1 Q_2 \frac{\partial}{\partial Y} P \gtrsim \left(Q_1^2 \pm Q_2^2 \frac{\partial^2}{\partial Y^2}\right) P,$$

$$Q_1 Q_2 \frac{\partial^2}{\partial Y \partial Z} P \gtrsim \left(Q_1^2 \frac{\partial^2}{\partial Y^2} \pm Q_2^2 \frac{\partial^2}{\partial Z^2}\right) P, \text{ etc.}$$

By these formulae the different terms of the wave equation can be compared. In particular, it follows from (8.5) that

$$\lambda Q_1 Q_2 \ll Q_1^2 \pm Q_2^2, \text{ provided } \lambda \ll 1 \quad (8.6)$$

another important formula which will be largely used to simplify the wave equation.

9. General approximate wave equation

If we solve the first three equations (6.4) for $D'x$, $D'Y$ and $D'Z$ we get, in the case of a wave, the following symmetrical expressions:

$$\left.\begin{aligned}
D'x &= \frac{i}{G'}\left[-\left(1 + \frac{F' r \omega_Y}{N^2}\right) k + \right. \\
&\left. + \frac{F'}{N}\left(\frac{\partial}{\partial Y} + s'_Y\right) - \frac{F_2 N}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\right] \partial' p, \\
D'Y &= \frac{1}{G'}\left[\frac{F'}{N} k - (1 + \varepsilon')\left(\frac{\partial}{\partial Y} + s'_Y\right) - \right. \\
&\left. - \frac{F_2 F_1'}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\right] \partial' p, \\
D'Z &= \frac{1}{G'}\left[\frac{F_2 N}{N_1^2 - N^2} k - \frac{F_2 F_1'}{N_1^2 - N^2} \cdot \right. \\
&\left. \cdot \left(\frac{\partial}{\partial Y} + s'_Y\right) - \frac{G_1}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\right] \partial' p.
\end{aligned}\right\} \quad (9.1)$$

(see (3.h) for symbols).

If we substitute from (9.1) in the last equation (6.4) and observe that, according to (7.1), we have

$$k_Y / k = - r_Y / r;\ N_Y = k r \omega_Y \quad (9.2)$$

we obtain, for the wave equation,

$$(A_1 + A_2 + A_3 + A_4)\ \partial' p = 0 \quad (9.3)$$

with

$$\left.\begin{aligned}
A_1 &= -\frac{k^2}{G'} + \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right) \frac{1 + \varepsilon'}{G} \cdot \\
&\cdot \left(\frac{\partial}{\partial Y} + s'_Y\right) + \left(\frac{\partial}{\partial Z} - s'_Z + \Phi_Y\right) \cdot \\
&\cdot \frac{G_1}{G'(N_1^2 - N^2)}\left(\frac{\partial}{\partial Z} + s'_Z\right), \\
A_2 &= \frac{F' k}{G' N}\left(\frac{\partial}{\partial Y} + s'_Y\right) - \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right) \cdot \\
&\cdot \frac{F' k}{G' N} - \frac{1}{G'} \frac{F' r \omega_Y}{N^2} k^2 = \\
&= \frac{F' k}{G' N}\left(\frac{G'_Y}{G'} - \frac{F'_Y}{F'} + 2 s'_Y + \frac{r_Y}{r} + \frac{s_{ZY}}{s_Z}\right), \\
A_3 &= \frac{F_2 N k}{G'(N_1^2 - N^2)}\left(\frac{\partial}{\partial Z} + s'_Z\right) - \\
&- \left(\frac{\partial}{\partial Z} - s'_Z + \Phi_Y\right) \frac{F_2 N k}{G'(N_1^2 - N^2)} \\
A_4 &= \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right) \frac{F_2 F_1'}{G'(N_1^2 - N^2)} \cdot \\
&\cdot \left(\frac{\partial}{\partial Z} - s'_Z\right) - \left(\frac{\partial}{\partial Z} - s'_Z + \Phi_Y\right) \cdot \\
&\cdot \frac{F_2 F_1'}{G'(N_1^2 - N^2)}\left(\frac{\partial}{\partial Y} + s'_Y\right)
\end{aligned}\right\}$$

Let us note here the important result that the term in k^2/N^2 is exactly cancelled out in the expression for A_2 in virtue of (9.2), at the same time as the terms in $\partial/\partial Y$ cancel out each other.

Let us now refer to the current values of the parameters, as observed in the meteorological atmosphere. We have in cgs units

$$\left.\begin{array}{l} s_Z \sim 10^{-7};\ -p_Z/q \# g \# 10^3;\ F_1 \# f \gtrsim 10^{-4}; \\ F_2 \gtrsim 10^{-4};\ a \sim 10^{-9};\ r\omega_Y \text{ and } u_Y \gtrsim f;\ r\omega_Z \text{ and} \\ u_Z \gtrsim 10^{-3};\ u \gtrsim 10^4;\ 1/R \sim 10^{-9};\ g_y/g \gtrsim 10^{-11} \end{array}\right\} \tag{9.4.1}$$

whence

$$\left.\begin{array}{l} F_1' \# F - u_Y \sim f;\ s_y \gtrsim 10^{-10};\ \varepsilon \gtrsim 10\, f \ll 1; \\ \varepsilon' \ll 1;\ s_Z' \sim 10^{-6};\ s_z \# s_Z;\ N_1 \# N_0 \sim 10^{-2}; \\ F' \# F_1 \# F;\ G' \# G_1 \# G;\ \varphi_Y \ll s_Z' \end{array}\right\} \tag{9.4.2}$$

Using these relations, we can prove that, in the wave equation,

i) G_1 and F_1 may be substituted everywhere to G' and F' respectively, and ε' may be omitted in the second term of A_1. This is obvious in virtue of (8.3) where one makes successively $Q_0 = G_1$ and $Q = G'$; $Q_0 = F_1$ and $Q = F'$; $Q_0 = 1$ and $Q = 1 + \varepsilon'$.

ii) G_{1Y} and F_{1Y} may be likewise substituted to G_Y' and F_Y' in the second expression for A_2.

We have to prove that we may neglect the two quantities

$$a_1 = \frac{1}{G'}\frac{F'k}{G'N}(G' - G_1)_Y;$$

$$a_2 = \frac{1}{G'}\frac{k}{N}(F' - F_1)_Y$$

From the definitions of G' and F' we get, since $kr\omega_Y = N_Y \gtrsim N(\partial/\partial Y)$,

$$a_1 \gtrsim \frac{1}{G'}\frac{F'k}{G'N}(G' - G_1)\frac{\partial}{\partial Y} \gtrsim \frac{\varepsilon'}{G'} \cdot \left(\frac{F'F_1'}{G'}\frac{\partial^2}{\partial Y^2} - \frac{F'N}{G'}k\frac{\partial}{\partial Y}\right);$$

$$a_2 \gtrsim \frac{1}{G'}\frac{k}{N}(F' - F_1)\frac{\partial}{\partial Y} \gtrsim \frac{\varepsilon'}{G'}\frac{\partial^2}{\partial Y^2}$$

whence, since $F'F_1' \gtrsim G'$; $F'N \gtrsim G'$; $\varepsilon' \ll 1$, and according to (8.5),

$$a_1 \ll \frac{1}{G'}\left(\frac{\partial^2}{\partial Y^2} - k\frac{\partial}{\partial Y}\right) \gtrsim \frac{1}{G'}\left(k^2 + \frac{\partial^2}{\partial Y^2}\right) \gtrsim A_1;$$

$$a_2 \ll \frac{1}{G'}\frac{\partial^2}{\partial Y^2} \gtrsim A_1$$

iii) Φ_Y may be omitted everywhere.

Owing to the relations $\Phi_Y = \varphi_Y + \varepsilon_Y$; $\varphi_Y \ll s_Z'$; $\varepsilon_Y \gtrsim \varepsilon\ (\partial/\partial Y)$, we have only to prove that we may neglect the three quantities

$$\frac{\varepsilon}{N_1^2 - N^2}\frac{\partial}{\partial Y}\left(\frac{\partial}{\partial Z} + s_Z'\right);\ \frac{F_2 N\varepsilon}{G'(N_1^2 - N^2)}k\frac{\partial}{\partial Y};$$

$$\frac{F_2F_1'\varepsilon}{G'(N_1^2 - N^2)}\frac{\partial}{\partial Y}\left(\frac{\partial}{\partial Y} + s_Y'\right)$$

Since we have, by (8.5),

$$A_1 \gtrsim \frac{1}{G'}k^2 + \frac{1}{G'}\left(\frac{\partial}{\partial Y} \pm s_Y'\right)^2 + \frac{1}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} \pm s_Z'\right)^2$$

it follows from (8.6) that these terms are negligible compared to A_1 provided

$$\varepsilon\sqrt{\frac{G'}{N_1^2 - N^2}} \ll 1;\ \varepsilon\frac{F_2N}{N_1^2 - N^2} \ll 1;\ \varepsilon\frac{F_2F_1'}{N_1^2 - N^2} \ll 1$$

and these three conditions are actually satisfied.

iv) The remaining terms of A_3 and A_4 are negligible compared to A_1. Since $s_{ZY}/s_Z \gtrsim \gtrsim \partial/\partial Y$, this is readily proved by putting in (8.6), successively,

$$Q_1 = \frac{1}{\sqrt{G'}}k;\ Q_2 = \frac{1}{\sqrt{N_1^2 - N^2}}\left(\frac{\partial}{\partial Z} \pm s_Z'\right);$$

$$\lambda = \frac{F_2N}{\sqrt{G'(N_1^2 - N^2)}}$$

$$Q_1 = \frac{1}{\sqrt{G'}}\left(\frac{\partial}{\partial Y} \pm s_Y'\right);\ Q_2 = \frac{1}{\sqrt{N_1^2 - N^2}} \cdot \left(\frac{\partial}{\partial Z} \pm s_Z'\right);\ \lambda = \frac{F_2F_1'}{\sqrt{G'(N_1^2 - N^2)}}$$

for in both cases we have $\lambda \ll 1$.

v) N_0 may be substituted to N_1 in the expression for A_1. Let us put

$$C_1 = \left(\frac{\partial}{\partial Z} - s_Z'\right)\frac{1}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s_Z'\right);$$

$$C_0 = \left(\frac{\partial}{\partial Z} - s_Z'\right)\frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s_Z'\right);$$

$$\frac{1}{N_1^2 - N^2} - \frac{1 + \varepsilon_1}{N_0^2 - N^2}\ (\varepsilon_1 \ll 1).$$

If we refer to the definitions of N_0 and N_1 and to equations of motion, we find that

$$\varepsilon_1 = \varepsilon_2 + \mu(\omega^2 - \omega_0^2), \text{ with}$$

$$\mu = \frac{s_Z r \cos \Phi}{N_1^2 - N^2}; \quad \varepsilon_2 = -\frac{F_2 r \omega_Z}{N_1^2 - N^2}$$

whence, by (8.2),

$$C_1 - C_0 \sim \mu \left(\frac{\partial}{\partial Z} - s_Z'\right) \frac{\omega^2 - \omega_0^2}{N_0^2 - N^2} \cdot$$

$$\cdot \left(\frac{\partial}{\partial Z} - s_Z'\right) + \varepsilon_2 C_0 - \varepsilon_1 C_0 -$$

$$- \mu \frac{(\omega^2 - \omega_0^2)_Z}{N_0^2 - N^2} \left(\frac{\partial}{\partial Z} + s_Z'\right) - \varepsilon_1 C_0 -$$

$$- \varepsilon_2 \frac{s_Z}{s_Z'} \frac{s_Z'}{N_0^2 - N^2} \left(\frac{\partial}{\partial Z} + s_Z'\right) \gtrsim \varepsilon_1 C_0 -$$

$$- \varepsilon_2 \frac{s_Z}{s_Z'} C_0 \ll C_0$$

in other words, $C_1 \# C_0$.

We thus arrive at the following simplified wave equation:

$$\left[\frac{k^2}{G_1} - \left(\frac{\partial}{\partial Y} - s_Y' - \frac{s_{ZY}}{s_Z}\right) \frac{1}{G_1} \left(\frac{\partial}{\partial Y} + s_Y'\right) - \right.$$

$$- \left(\frac{\partial}{\partial Z} - s_Z'\right) \frac{1}{N_0^2 - N^2} \left(\frac{\partial}{\partial Z} + s_Z'\right) + a -$$

$$\left. - \frac{k}{G_1 N} B_1\right] \partial' p = 0 \qquad (9.5)$$

with

$$B_1 = F_1 \left(\frac{G_{1Y}}{G_1} - \frac{F_{1Y}}{F_1} + 2 s_Y' + \frac{r_Y}{r} + \frac{s_{ZY}}{s_Z}\right)$$

This equation can be transformed in several ways.

We can first prove, by the same methods used for above simplifications, that G_1, F_1, s_Z, s_Z' and $\partial/\partial Z$ may be replaced everywhere with G, F, s_z, s_z' and $\partial/\partial z$, respectively. That means that we may disregard the small departure of Z-direction from the vertical, as we could expect it from the smallness of all the horizontal gradients compared to the vertical ones, and that furthermore $r\omega_Y$ may be replaced with u_Y.

On the other hand, if we refer to the definition of s' and to relation (6.3) we get, since $s_Y = 0$,

$$2 s_Y' + \frac{r_Y}{r} - m \frac{p_Y}{p} = -a \frac{p_Y}{q} =$$

$$= \frac{a}{s_Z} (F_1 r \omega_Z + F_2 r \omega_Y)$$

and, according to (9.2),

$$G_Y = F(F - u_Y)_Y + F_Y(F - u_Y) - 2 Nkr\omega_Y$$

whence we get, after a few obvious simplifications have been made, the following form for the wave equation:

$$\left[\left(1 + \frac{2 F u_Y}{G}\right) \frac{k^2}{G} - \left(\frac{\partial}{\partial Y} - s_Y' - \frac{s_{zY}}{s_z}\right) \frac{1}{G} \cdot \right.$$

$$\cdot \left(\frac{\partial}{\partial Y} + s_Y'\right) - \left(\frac{\partial}{\partial z} - s_z'\right) \frac{1}{N_0^2 - N^2} \cdot$$

$$\left. \cdot \left(\frac{\partial}{\partial z} + s_z'\right) + a - \frac{k}{GN} B\right] \partial' p = 0$$

(9.6)

where B is the quantity defined at the end of (3.h).

Another practically equivalent form is obtained by introducing the variable $\partial'' p$ and the parameter s'' [see (3.f) and (3.e)], as follows:

$$\left[\left(1 + \frac{2 F u_Y}{G}\right) \frac{k^2}{G} - \left(\frac{\partial}{\partial Y} - s_Y''\right) \frac{1}{G} \cdot \right.$$

$$\cdot \left(\frac{\partial}{\partial Y} + s_Y''\right) - \frac{1}{N_0^2} \left(\frac{\partial}{\partial z} - s_z''\right) \frac{N_0^2}{N_0^2 - N^2} \cdot$$

$$\left. \cdot \left(\frac{\partial}{\partial z} + s_z''\right) + a - \frac{k}{GN} B\right] \partial'' p = 0$$

(9.7)

[In (9.6) and (9.7) $r\omega_Y$ may be substituted to u_Y, and the same substitution may be made in the expressions for G and B.]

We observe that *the vector* $2\,\mathcal{W}$ *only figures by its vertical component* in the wave equation (9.6) or (9.7), in agreement with the current assertion that the horizontal component of earth's rotation may be always neglected in atmospheric motions.

Let us now compare the orders of magnitude of the different terms of the wave equation (9.5). We have, when dealing with operators applied to $\partial' p$,

$$\frac{k^2}{G_1} - \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right)\frac{1}{G_1}\left(\frac{\partial}{\partial Y} + s'_Y\right) \sim$$

$$\sim \frac{1}{G}\left[k^2 + \left(\frac{\partial}{\partial Y} + s'_Y\right)^2\right],$$

$$\left(\frac{\partial}{\partial Z} - s'_Z\right)\frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right) \sim$$

$$\sim \frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)^2,$$

$$a \gtrsim \frac{s'^2_Z}{N_0^2} \gtrsim \frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)^2,$$

$$\frac{k}{G_1 N} B_1 \gtrsim \frac{F}{N}\frac{1}{G}\, k \left(\frac{\partial}{\partial Y} + s'_Y\right) \gtrsim$$

$$\gtrsim \frac{F}{N}\frac{1}{G}\left[k^2 + \left(\frac{\partial}{\partial Y} + s'_Y\right)^2\right],$$

whence, by (9.5),

$$\frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)^2 \gtrsim \left(1 + \frac{F}{N}\right)\frac{1}{G} \cdot$$

$$\cdot \left[k^2 + \left(\frac{\partial}{\partial Y} + s'_Y\right)^2\right] \gtrsim \left(1 + \frac{F}{N}\right)^2 \cdot$$

$$\cdot \frac{1}{G}\left(k + \frac{\partial}{\partial Y} + s'_Y\right)^2$$

and therefore

$$\left(\frac{\partial}{\partial Z} + s'_Z\right)\partial' p \gtrsim \sqrt{\frac{N_0^2 - N^2}{G}}\left(1 + \frac{F}{N}\right) \cdot$$

$$\cdot \left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p \qquad (9.8)$$

This formula will be very useful in the next sections.

10. General approximate formulae for the vibration components

In the case of a wave, the vector $D\mathbf{M}$ may be conveniently called the *vibration*, and its components are given as functions of $\partial' p$ by formulae (9.1). Let us now show that these formulae can be simplified like the wave equation, *as far as the magnitude and direction of the vibration is concerned*. For this purpose, we must observe that any term of the expression for $D'x$, $D'Y$ or $D'Z$ may be omitted if it is small compared to any one of these three components.

i) The last terms of $D'x$ and $D'Y$ are negligible compared to the other terms of these components.

This is readily seen by using relation (9.8), whence we deduce that

$$\frac{F_2 N}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\partial' p \gtrsim \frac{F_2 N}{\sqrt{G(N_0^2 - N^2)}} \cdot$$

$$\cdot \left(1 + \frac{F'}{N}\right)\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p \lessapprox$$

$$\lessapprox \left(1 + \frac{F'}{N}\right)\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p$$

$$\frac{F_2 F_1'}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\partial' p \gtrsim \frac{F_2 F_1'}{\sqrt{G(N_0^2 - N^2)}} \cdot$$

$$\cdot \left(1 + \frac{F'}{N}\right)\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p \lessapprox$$

$$\lessapprox \left(1 + \frac{F'}{N}\right)\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p$$

ii) The first two terms of $D'Z$ are negligible compared to the first term of $D'x$ and the second term of $D'Y$, respectively.

This results from the fact that

$$\frac{F_2 N}{N_1^2 - N^2} \lessapprox 1 \text{ and } \frac{F_2 F_1'}{N_1^2 - N^2} \lessapprox 1$$

iii) $\partial/\partial Z$ and s'_z may be replaced with $\partial/\partial z$ and s'_Z, respectively, in the last term of $D'Z$.

Similar proof as above.

iv) $D'y$ and $D'z$ may be computed by the same formulae as $D'Y$ and $D'Z$, respectively.

This is obvious from the relations

$$D'y = D'Y \cos\varepsilon + D'Z \sin\varepsilon;$$
$$D'z = D'Z \cos\varepsilon - D'Y \sin\varepsilon$$

Finally, if we simplify the terms of (9.1)

according to (9.4), we arrive at the following approximate formulae:

$$\left.\begin{aligned} D'x &= \frac{i}{G}\left[-\left(1+\frac{Fr\omega_Y}{N^2}\right)k + \right. \\ &\left. + \frac{F}{N}\left(\frac{\partial}{\partial Y}+s'_Y\right)\right]\partial' p \\ D'Y \text{ or } D'y &= \frac{1}{G}\left[\frac{F}{N}k-\left(\frac{\partial}{\partial Y}+s'_Y\right)\right]\partial' p \\ D'Z \text{ or } D'z &= -\frac{1}{N_0^2-N^2}\left(\frac{\partial}{\partial Z}+s'_Z\right)\cdot \\ &\cdot \partial' p \text{ or } -\frac{1}{N_0^2-N^2}\left(\frac{\partial}{\partial_z}+s_z'\right)\partial' p \end{aligned}\right\} \quad (10.1)$$

We observe that, just as in the wave equation, the vector $2\,\omega$ only figures by its vertical component.

11. Classification of waves according to orbital period P

Up to this point, we have made no assumption about N. We shall now see that further simplifications appear if this variable is assumed to have a definite order of magnitude compared to the parameters N_0 and f.

Since we have always $N_0/f \gtrsim 10^2$, it is obvious that we are always in only one of the three following cases:

i) $N \gtrsim N_0 \gg f$ ($P < 2$ hours), short-period waves

ii) $N \ll N_0$ and $\gtrsim f$ (2 hours $< P <$ 5 pendulum days), middle-period waves

iii) $N \ll f \ll N_0$ ($P >$ 5 pendulum days), long-period waves

According to these definitions, the width of the middle-period waves interval in the frequency spectrum becomes larger and larger as latitude decreases, and at the equator there are no long-period waves at all.

i) Short-period waves

From the assumption $N \gg f$, it can be proved, by the same method used in preceding sections, that the wave equation reduces to

$$\left[\frac{k^2}{N^2}-\left(\frac{\partial}{\partial y}-s_y'\right)\frac{1}{N^2}\left(\frac{\partial}{\partial y}+s_y'\right)- \right.$$
$$\left. -\left(\frac{\partial}{\partial z}-s_z'\right)\frac{1}{N^2-N_0^2}\left(\frac{\partial}{\partial z}+s_z'\right)-a\right]\partial' p = 0 \quad (11.1)$$

On the other hand, the expressions (10.1) for the vibration components may be replaced with

$$D'x = \frac{i}{N^2}\,k\,\partial' p;\; D'y = \frac{1}{N^2}\left(\frac{\partial}{\partial y}+s_y'\right)\partial' p;$$
$$D'z = \frac{1}{N^2-N_0^2}\left(\frac{\partial}{\partial z}+s_z'\right)\partial' p \quad (11.2)$$

We can see that now neither the vector ω nor its ascendant figure in the formulae, so that we can obtain the same formulae by assuming from the beginning that $\omega = 0$.

ii) Middle-period waves

If $N \ll N_0$, we can simplify the wave equation (9.6) or (9.7) by omitting N^2 beside N_0^2, but a more interesting result concerns the vibration components.

The relation (9.8) gives

$$\frac{1}{N_0^2-N^2}\left(\frac{\partial}{\partial Z}+s'_Z\right)\partial' p \lesssim \frac{1}{N_0\sqrt{G}}\cdot$$
$$\cdot\left(1+\frac{F}{N}\right)\left(k+\frac{\partial}{\partial Y}+s'_Y\right)\partial' p \ll \frac{1}{G}\cdot$$
$$\cdot\left(1+\frac{F}{N}\right)\left(k+\frac{\partial}{\partial Y}+s'_Y\right)\partial' p$$

whence we conclude, by (10.1), that

$$D'z \ll D'x \text{ or } D'y$$

In other words, the vertical component of the motion is negligible compared to its horizontal components. This is in agreement with the observed fact that if the scale of an atmospheric motion is such that its vertical acceleration is negligible (an assumption which corresponds to the condition $N \ll N_0$) this motion is always mainly horizontal.

Let us observe here that, in low and middle latitudes, some of the middle-period waves satisfy both of the conditions $N \ll N_0$ and

$N \gg f$, so that the special properties of the short-period waves also belong to these waves.

iii) Long-period waves

For these waves, we have of course the same fundamental property as for the middle-period ones, namely the predominance of the horizontal components of the motion, but further simplifications result from the condition $N \ll f$:

In first place the expressions for G and B reduce to

$$G = F(F - u_Y);$$

$$B = F\left[\frac{(F-u_Y)_Y}{F-u_Y} + \frac{aFu_z + s_{zY}}{s_z}\right]$$

and secondly the horizontal components of the vibration may be computed by the approximate formulae

$$D'x = \frac{i}{F-u_Y}\frac{I}{N}\left[-\frac{r\omega_Y}{N}k + \left(\frac{\partial}{\partial Y} + s'_Y\right)\right]\partial' p =$$

$$= \frac{i}{F-u_Y}\left(\frac{\partial}{\partial Y} + s'_Y\right)\frac{1}{N}\partial' p;$$

$$D'y = \frac{1}{F-u_Y}\frac{k}{N}\partial' p$$

12. Wave equation with constant coefficients

a) Let us first consider the case of a short-period wave, with

$$\left.\begin{array}{l}\omega = \text{const, whence } N = \text{const} \\ N_0 = \text{const};\ \partial/\partial y \lesssim 10^{-8};\ \partial/\partial z \lesssim 10^{-5}\end{array}\right\} \quad (12.1)$$

We thus have $s_y' \ll \partial/\partial y;\ s_z' \ll \partial/\partial z$

so that the wave equation (11.1) reduces to

$$\left(\frac{k^2}{N^2} - \frac{1}{N^2}\frac{\partial^2}{\partial y^2} - \frac{1}{N^2 - N_0^2}\frac{\partial^2}{\partial z^2} - a\right)\partial' p = 0 \quad (12.2)$$

and all its coefficients may be considered as constants, as well as r and R, since their scale of space-distribution is, according to (12.1), large compared to that of the perturbation. We may therefore take, as particular solutions, the pseudo-plane waves defined by

$$\partial' p = \partial' p_1 \exp.\ i\ (k\ x + k'y + k''z - n't) \quad (12.3)$$

where $x = rL$; $y = R\varphi$; z is the altitude, and $\partial' p_1$, k', k'' are three constants.

From (12.2) it follows that k, k', k'' and N must be connected together by the relation

$$(k^2 + k'^2)/N^2 + k''^2/(N^2 - N_0^2) - a = 0$$

or

$$a\,N^4 - (K^2 + a\,N_0^2)\,N^2 + N_0^2\,K^2 \sin^2\psi = 0$$

where $K = \sqrt{k^2 + k'^2 + k''^2}$ is the geometrical wave-number, and ψ is the slope angle of the wave surface ($\tan\psi = k''/K$).

a.1) If k' and k'' are real (internal waves with uniform amplitude), it results from (12.1) that $K \lesssim 10^{-5}$, whence $a\,N_0^2 \ll K^2$ and $\sqrt{a}N_0 \sin\psi \ll K$, so that the above relation may be written as well

$$(a\,N^2 - K^2)\ (N^2 - N_0^2 \sin^2\psi) = 0$$

We thus get two types of waves:

i) If $aN^2 - K^2 = 0$, the orbital phase-velocity $U = |N/k|$ has the uniform value $U = 1/\sqrt{a}$, the sound velocity, and since $N \gg N_0$ the relations (11.2) give us

$$\frac{Dx}{k} = \frac{Dy}{k'} = \frac{Dz}{k''}$$

a proportion which means that the vibration is *longitudinal*. The corresponding waves are obviously nothing but the classical *elastic waves*.

ii) If $N^2 - N_0^2 \sin^2\psi = 0$, whence $|N| = N_0 \sin\psi$, we get from (11.2)

$$kDx + k'Dy + k''Dz = 0$$

in other words the vibration is *transversal*. We have proposed the name *internal gravity-waves* for the corresponding waves.

a.2) If k'' is a pure imaginary number, k' being real, the waves are no longer internal waves with uniform amplitude,

they are surface waves with an amplitude which decreases exponentially upwards or downwards. For instance if we assume $N \gg N_0$, and $aN^2 \ll 10^{-10}$, we get $k''^2 = -(k^2 + k'^2)$, and the corresponding waves are of the type of the classical gravitational waves observed near the surface of deep water for instance.

b) Let us then consider the case of a long-period wave, with

$$\left.\begin{array}{l}\omega = \text{const, whence again } N = \text{const} \\ N_0 = \text{const};\ k \lesssim 10^{-8};\ \partial/\partial y \sim 10^{-9}\end{array}\right\} \quad (12.4)$$

Since $\omega = \text{const}$ and $s_z \neq \text{const}$, we have

$$G = F^2;\ B \neq F_Y \neq f_y$$

and the wave equation (9.6) reduces to

$$\left(k^2 - \frac{F^2}{N_0^2}\frac{\partial^2}{\partial z^2} - \frac{k}{N} f_y\right) \partial' p = 0$$

If the study of the motion is restricted to a narrow middle-latitude zone of say 20° or 30° in width, all the coefficients of this equation may be taken as constants, as well as r and R. We may therefore again consider particular solutions of the form (12.3), with now $k' = 0$, and

$$k^2 + (F^2/N_0^2)\, k''^2 - (k/N) f_y = 0$$

If we consider the orbital phase-velocity $U_x = -(N/k)$ in the x-direction, we get from this relation

$$U_x = -f_y/[k^2 + (F^2/N_0^2)\, k''^2] \quad (12.4)$$

and if, as a result of proper boundary conditions, the term in k'' is small compared to the term in k, this dispersion equation reduces to ROSSBY's formula for long atmospheric waves.

c) If in (b) we replace the condition $\omega = \text{const}$ with the two following ones, which are practically compatible:

$$\text{absolute vertical vorticity} = F - u_y = \text{const};\quad u_z = 0$$

we have $F - u_Y = \text{const}$, whence $B = 0$, and the wave equation becomes

$$\left[\frac{F + u_Y}{F} k^2 - \left(\frac{F - u_Y}{N_0}\right)^2 \frac{\partial^2}{\partial z^2}\right] \partial' p = 0$$

If the study of the motion is again restricted to a narrow zone, the coefficients of this equation may be taken as constants, and we may again consider solutions of the form (12.3), with $k' = 0$ and

$$k'' = \pm i \sqrt{\frac{F + u_Y}{F}}\, \frac{N_0}{F - u_Y}\, k$$

13. Considerations about some incorrect results recently published

a) Some authors have considered that, since the large-scale motions are always mainly horizontal, it is permissible, in the case of long-period waves, to disregard the vertical motion in the perturbation equations, a procedure which obviously facilitates the derivation of the wave equation. It may be useful to show how far the wave equation obtained in this fashion is incorrect.

If we neglect the vertical motion, we are led to replace the perturbation equations (6.4) with the following approximate ones, referred to x, y, z-axes:

$$\left.\begin{array}{l}\dfrac{D^2}{Dt^2} D'x - F \dfrac{D}{Dt} D'y + \dfrac{\partial}{\partial x} \partial' p = 0, \\ F \dfrac{D}{Dt} D'x + \left(\dfrac{D^2}{Dt^2} - Fr\omega_y + fus_y\right) D'y + \\ \qquad + \left(\dfrac{\partial}{\partial y} + s_y'\right) \partial' p = 0, \\ \dfrac{\partial}{\partial x} D'x + \left(\dfrac{\partial}{\partial y} - s'_y\right) D'y + a\partial' p = 0.\end{array}\right\}$$

whence, by eliminating $D'x$ and $D'y$, we easily get the wave equation

$$\left[\left(1 - \frac{fus_y}{N^2}\right)\frac{k^2}{G_0} - \left(\frac{\partial}{\partial y} - s_y'\right)\frac{1}{G_0} \cdot \left(\frac{\partial}{\partial y} + s_y'\right) + a - \frac{k}{G_0 N} B_0\right] \partial' p = 0 \quad (13.1)$$

with

$$G_0 = G + fus_y$$

$$B_0 = F\left(\frac{G_{0y}}{G_0} - \frac{F_y}{F} + 2\, s_y' + \frac{r_y}{r}\right) \neq$$

$$\neq F\left(\frac{G_{0y}}{G_0} - \frac{F_y}{F} + afu - s_y\right)$$

Let us now consider the special case where

$$\omega_y = 0, \text{ whence } N_y = 0, \text{ and}$$
$$N^2 \ll F^2;\ k \gtrsim 10^{-8};\ \partial/\partial y \sim 10^{-9}$$

With these assumptions, (13.1) reduces to the following dispersion equation:

$$\left(1 - \frac{fus_y}{N^2}\right) k^2 - \frac{k}{N} B_0 = 0$$

with now

$$B_0 \# f_y + f(afu - s_y)$$

This is a second-degree equation in N, the roots of which are imaginary if

$$fus_y < 0 \text{ and } k^2 > \frac{B_0^2}{4\,|fus_y|}$$

so that the above derivation leads to unstable waves in a westerly current if the potential temperature decreases towards the pole. It is clear that this result, which has been given as an explanation for the origin of some of the extra-tropical atmospheric waves, is due to the presence, in wave equation (13.1), of the term in k^2/N^2, which has not been cancelled out in the derivation of this equation, precisely because the vertical component of the motion has been neglected. Since such a term does not exist in the correct wave equation, as already emphasized, we must conclude that the existence of unstable waves in the westerlies cannot be justified in this simple fashion.

By this example, we also understand to what kind of errors may lead the application of the method under discussion.

b) Another simplification which is frequently made is that, according to what is assumed in the so-called parcel method in the theory of hydrostatic stability, the local change of pressure ∂p is negligible, so that one may put $\partial p = 0$ (or $\partial' p = 0$) in the wave equations.

If we do that in equations (6.4), and if we only retain the first three of these equations, we get, by eliminating the unknowns $D'x$, $D'Y$, $D'Z$, the following condition in the case of a wave:

$$G \# F(F - r\omega_Y) - N^2 = 0 \quad (13.2)$$

which determines the value of N at any point.

The corresponding waves, in the assumption that they can exist, are nothing but the so-called *isobaric waves*, so often considered in theoretical meteorology, and the condition (13.2) shows that these waves would be stable or unstable, according as $r\omega_Y$ is smaller or larger than F. If for instance $\omega = \text{const}$, we get $N = \pm F$, and if $\omega = \omega_0$ the orbital frequency must be equal to the Coriolis parameter f.

But it is easy to see that such waves cannot exist except for particular undisturbed motions. For N is already connected to ω by the relation $N = n\omega - n'$, which is compatible with (13.2) only for a special space-distribution of ω. Furthermore, whatever may be this special distribution, the isobaric waves are necessarily stable, for $n\omega - n'$ can be a pure imaginary everywhere only if ω is a constant, and we have already seen that in this case N is real.

We can therefore conclude that the consideration of isobaric waves is only of poor interest in the case of a distorted current and cannot explain the apparition of unstable waves.

14. Conclusion

This paper was written with the intent, not to give the solution of a particular perturbation-problem related to a zonal air-current, but only to secure the first step in the way to such a solution, namely the derivation of the wave equation. The perturbation problem is thus reduced to a purely mathematical one: the resolution of the wave equation under proper boundary conditions.

This resolution is certainly not an easy task for most of practical problems, but it is clear that it will be always possible to manage it, by using modern computers if necessary, and we have proved in the preceding section that, in fact, most of the disagreements between the results obtained by various authors are to be imputed to incorrect derivations of the wave equation.

We think it therefore not superfluous, in view of generalizations to non-zonal or non-adiabatic motions for instance, to emphasize here that, when deriving a differential equa-

tion, *the simplifications resulting from the order of magnitude of the parameters must be made only at the very end of the derivation, and not in the course of the computations.* According to this principle, the approximate meteorological laws generally applied to large-scale motions (vanishing of vertical acceleration or velocity, vorticity theorem or conservation of vertical vorticity, etc.), though very useful to give the distributions of some of the hydrodynamical quantities when some others are known, must never be used as starting relations in a sequence of computations where differentiations are involved.

REFERENCES

Bjerknes, V. and collaborators, 1933: Chap. 14, Physikalische Hydrodynamik, Jul. Springer, Berlin 1933; Hydrodynamique Physique, Presses Univ. de France, Paris 1934.

Charney, J. G., 1947: The Dynamics of Long Waves in a Baroclinic Westerly Current, *J. Met.*, **4**, No 5.

Eady, E. T., 1949: Long Waves and Cyclone Waves, *Tellus*, **1**, No 3.

Fjoertoft, R., 1941: On the Deepening of a Polar Front Cyclone, *Met. Ann. Oslo*, No 1.

Fjoertoft, R., 1946: On the Frontogenesis and Cyclogenesis in the Atmosphere, Part I: On the Stability of the Stationary Circular Vortex, *Geof. Publ.*, **16**, No 5.

Helmholtz, 1868: Ueber discontinuierliche Flüssigkeitsbewegungen, *Berl. Monatsber.*, April; *Phil. Mag.*, Nov.

Hoeiland, E., 1939: On the Interpretation and Application of the Circulation Theorems of V. Bjerknes, *Archiv for mathem. og naturv., Oslo*, **42**, No 5.

—, 1941: On the Stability of the Circular Vortex, *Avh. utgitt av Det Norske Vid.-Ak., Oslo, Matem. Nat. Klass, Oslo*, No 11.

Jaw, J., 1946: The Formation of the Semi-Permanent Centers of Action in Relation to the Horizontal Solenoidal Field, *J. Met.*, **3**, No 4.

Jeffreys, H., 1925: On the Formation of Water Waves by Wind, *Proc. Roy. Soc.* A, **107**.

Kelvin (Thomson, W.), 1871: Hydrokinetic Solutions and Observations, *Phil. Mag.*, **42**; *Nature*, **5**.

Küttner, J., 1939: Moazagotl und Föhnwelle, *Beitr. Phys. fr. Atm.*, **25**.

Lamb, H., 1932: Atmospheric Waves, Chap. 10, §§ 309—316, Hydrodynamics, Cambr. Univ. Press 1932.

Van Mieghem, J., 1944: Forme intrinsèque du critère d'instabilité dynamique de E. Kleinschmidt. *Bull. Sc. Ac. Roy. Belg.*, 5e série, **30**, p. 19; Interprétations énergétiques du critère d'instabilité de Kleinschmidt, ibid, **31**, p. 345; Perturbations d'un courant atmosphérique permanent zonal. *Mém. Inst. Mét. Belg.*, **18**, p. 1.

—, 1946: Sur la Stabilité du Courant Géostrophique, *La Météorologie*, Paris; Contribution à l'Etude de la Cyclogénèse, *Mém. Inst. Mét. Belg.*, **23**.

Pekeris, C. L., 1937: Atmospheric Oscillations, *Proc. Roy. Soc.* A, **158**.

Rossby, C.-G., 1947: On the Distribution of Angular Velocity in a Gaseous Envelope under the Influence of Large Scale Mixing Processes, *Bull. Am. Met. Soc.*, **28**.

Solberg, H., 1928: Integrationen der Atmosphärischen Störungsgleichungen, *Geof. Publ.*, **5**, No 9.

—, 1936: Le Mouvement d'Inertie de l'Atmosphere Stable et son Rôle dans la Théorie des Cyclones, UGGI, Ass. Met., Edinburgh 1936.

Taylor, G. I., 1936: The Oscillations of the Atmosphere, *Proc. Roy. Soc.* A, **156**.

Sur la circulation transversale associée à un courant atmosphérique

Par JACQUES VAN MIEGHEM, Université de Bruxelles

(Manuscript received 14 December 1949)

Abstract

A criticism of the interpretation of the circulation theorem is followed by an investigation of the different factors which contribute to a transversal circulation in a horizontal current.

1. L'étude de la circulation transversale associée au «jet-stream» [5] a conduit aux notions de circulation directe et de circulation inverse, au sujet desquelles il faut, croyons-nous, faire certaines réserves. Une circulation est dite directe lorsque son sens est celui prescrit par les solénoïdes isobares-isostères; elle est dite inverse dans le cas contraire. On admet qu'une circulation directe résulte des solénoïdes et ne se maintient qu'aux dépens de ceux-ci, tandis qu'une circulation inverse est supposée engendrer des solénoïdes isobares-isostères. De plus, confondant les notions «sens de circulation» et «sens du cycle thermodynamique décrit par une particule d'air», on affirme qu'une circulation directe transforme de l'énergie potentielle en énergie cinétique, une circulation inverse, de l'énergie cinétique en énergie potentielle [3].

L'interprétation du théorème de la circulation de V. Bjerknes à la lumière d'un principe de causalité ne peut conduire qu'à des contradictions. En voici un exemple récent. D'après P. Raethjen [4], le gradient isobare γ_p de la température crée une circulation qui rétablit le gradient défini par le taux d'accroissement du vent suivant la verticale. Lorsque le gradient γ_p est trop fort (c'est-à-dire, supérieur au gradient isobare Γ_p défini par le taux d'accroissement du vent réel suivant la verticale), l'air chaud s'élève et l'air froid descend (circulation directe), les variations adiabatiques de la température de l'air qui en résultent, entraînent une diminution du gradient γ_p tant que la valeur limite Γ_p n'est pas atteinte (condition d'équilibre de Margules). Inversément, lorsque le gradient γ_p est trop faible ($\gamma_p < \Gamma_p$), l'air chaud descend et l'air froid s'élève (circulation inverse) jusqu'à ce que le gradient γ_p atteigne la valeur limite Γ_p. E. Palmén [3], par contre, prétend que les forts gradients de température que l'on rencontre dans la surface de 500 mb où le «jet stream» contourne un creux, sont dus à une circulation inverse susceptible de créer des gradients de cette importance, laissant sous-entendre que les faibles gradients thermiques dans les dorsales de la même surface isobare, doivent être attribués à une circulation directe.

Ainsi, pour P. Raethjen, les solénoïdes constituent l'agent moteur de la circulation, tandis que pour E. Palmén, la circulation est créatrice de solénoïdes. De telles contradictions sont inévitables dès que l'on fait appel à un principe de causalité. Qu'il me suffise de rappeler que les équations de la mécanique expriment des relations entre certaines grandeurs, mais qu'elles n'établissent aucun lien causal entre celles-ci.

2. Le théorème classique de V. Bjerknes exprime que dans un fluide barocline, la circulation absolue est une propriété non con-

servative du mouvement du fluide et cela par une relation entre le champ du mouvement et le champ des solénoïdes isobares-isostères et non pas par un lien causal entre ces deux champs. L'accélération de la circulation absolue le long d'une courbe fluide et le nombre des solénoïdes isobares-isostères, que cette courbe entoure, varient simultanément dans le même sens, sans que l'on puisse affirmer que les solénoïdes sont la cause ou la conséquence de la circulation. Enfin, le sens de cette circulation est univoquement déterminé (sens solénoïdal).

Dans un système de référence fixe par rapport à la terre, l'accélération de la circulation *relative* le long d'une courbe fermée fluide est égale à la différence de deux termes: le premier n'est autre que le nombre de solénoïdes isobares-isostères embrassés par cette courbe (c'est-à-dire l'accélération de la circulation *absolue* le long de cette courbe), le second dépend de la rotation de la terre et de la déformation du milieu fluide à l'intérieur de la courbe envisagée. Le sens de la circulation relative est déterminé par le signe de cette différence. Cette circulation peut donc avoir le sens solénoïdal ou le sens opposé, sans qu'il soit pour cela possible d'affirmer que la circulation relative résulte de l'effet différentiel des deux termes susmentionnés, ou inversément, que la circulation engendre cet effet par une création ou une destruction appropriée de solénoïdes isobares-isostères.

3. Pour étudier la circulation transversale associée à un courant horizontal en un point quelconque O de ce courant, il est commode d'adopter un trièdre trirectangle dextrogyre Oxyz, fixe par rapport à la terre, dont l'axe Ox coïncide avec la direction du courant et l'axe Oz, avec la verticale ascendante de O. Dans ce cas, l'axe Oy est orienté à gauche du courant et par conséquent, dans la troposphère moyenne et supérieure de l'hémisphère Nord, cet axe est dirigé, à la fois, vers les basses pressions et les basses températures.

Dérivons par rapport à y l'équation du mouvement suivant Oz et par rapport à z l'équation du mouvement suivant Oy; en soustrayant les relations ainsi obtenues, on trouve

$$rot_x\left(\frac{d\vec{v}}{dt}\right) = \frac{\partial}{\partial y}\left(\frac{dv_z}{dt}\right) - \frac{\partial}{\partial z}\left(\frac{dv_y}{dt}\right) =$$

$$- \frac{\partial \alpha}{\partial z}\frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial z} + 2\omega_z \frac{\partial v_x}{\partial z}$$

$$- 2\omega_x\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) + 2\omega_y \frac{\partial v_x}{\partial y}$$

$$+ 2v_x \frac{\partial \omega_y}{\partial y} - 2v_y \frac{\partial \omega_x}{\partial y} \ldots\ldots (1)$$

Comme d'habitude, p et α représentent la pression atmosphérique et le volume spécifique de l'air, v_x, v_y, v_z les composantes coordonnées de la vitesse $\vec{v}$ de l'air par rapport à la terre (on a: $v_x > 0$ et en O, $v_y = 0$) et ω_x, ω_y, ω_z les composantes coordonnées de la rotation de la terre (on a: $\omega_x \equiv \omega \cos\varphi \sin\beta$, $\omega_y \equiv \omega \cos\varphi \cos\beta$, $\omega_z \equiv \omega \sin\varphi$, où φ désigne la latitude et β l'angle du courant avec la direction W $\rightarrow$ E).

Ensuite, remarquons que

$$\frac{\partial \omega_x}{\partial y} = -\frac{\omega}{a}\sin\varphi \cos\beta \sin\beta \qquad \text{et}$$

$$\frac{\partial \omega_y}{\partial y} = -\frac{\omega}{a}\sin\varphi \cos^2\beta;$$

d'où,

$$2\left(v_x \frac{\partial \omega_y}{\partial y} - v_y \frac{\partial \omega_x}{\partial y}\right) = -\frac{2\omega}{a}\sin\varphi \cos\beta$$

$$(v_x \cos\beta - v_y \sin\beta) = -\frac{2\omega_z}{a} u \cos\beta \ldots (2)$$

u étant la composante zonale du courant $\vec{v}$ (en O, on a: $u = v_x \cos\beta$) et a, le rayon moyen de la terre.

Le nombre N_x de solénoïdes isobares-isostères, par unité d'aire, en O, dans le plan transversal de ce point, est donné par

$$N_x \equiv \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial z} - \frac{\partial \alpha}{\partial z}\frac{\partial p}{\partial y} \cong -\frac{g}{T}\left(\frac{\delta T}{\delta y}\right)_p =$$

$$- 2\omega_z \frac{\partial v_g}{\partial z} > 0 \ldots\ldots\ldots (3)$$

compte tenu de l'équation des gaz parfaits, de l'équation de la statique suivant la verticale z et de l'équation d'équilibre de Margules; dans (3), $\left(\frac{\delta T}{\delta y}\right)_p$ représente l'ascendant isobare $-\gamma_p$ de la température absolue T de l'air suivant la direction transversale y, g, la pesanteur et v_g, la vitesse géostrophique de l'air en O.

Enfin, substituons dans (1) les relations (2), (3) et l'équation de continuité, on obtient

$$\mathrm{rot}_x\left(\frac{d\vec{v}}{dt}\right) = 2\,\omega_z\left[\frac{\partial v_x}{\partial z} - \frac{\partial v_g}{\partial z}\right] + 2\omega_x\frac{\partial v_x}{\partial x} - \frac{2\omega_x}{\alpha}\frac{d\alpha}{dt} + 2\omega_y\frac{\partial v_x}{\partial y} - 2\,\omega_z\frac{\cos\beta}{a}\,u \ldots \quad (4)$$

au point O. Rappelons que $\frac{\partial v_g}{\partial z}$ représente le vent thermique en ce point.

L'accélération de la circulation (ou ce qui revient au même, la circulation de l'accélération) le long de la frontière d'une unité de surface dans le plan transversal yz est égale à la composante longitudinale $\mathrm{rot}_x\left(\frac{d\vec{v}}{dt}\right)$ de la rotationnelle de l'accélération $\frac{d\vec{v}}{dt}$ de l'air, au point de convergence de la surface considérée. La formule (4) exprime donc l'accélération de la circulation, par unité de surface, en un point quelconque O, dans le plan transversal de ce point; elle s'applique aussi bien à un courant à composante $W\left(u > o, -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$ qu'à un courant à composante

$$E\left(u < o, \frac{\pi}{2} < \beta < \frac{3\,\pi}{2}\right).$$

Comme dans la troposphère moyenne et supérieure de l'hémisphère Nord, l'air froid est à gauche et l'air chaud à droite du courant $\vec{v}$, il se produira une circulation transversale relative *directe* lorsque $\mathrm{rot}_x\left(\frac{d\vec{v}}{dt}\right) < o$ (sens solénoïdal) et une circulation *inverse* lorsque $\mathrm{rot}_x\left(\frac{d\vec{v}}{dt}\right) > o$.

Examinons maintenant la contribution de chacun des cinq termes du second membre de (4) au sens de la circulation transversale relative.

1° Lorsqu'en un point quelconque O, le taux d'accroissement $\frac{\partial v_x}{\partial z}$ du courant v_x en ce point, suivant la verticale z (vertical wind shear), est inférieur (*supérieur*) au vent thermique $\frac{\partial v_g}{\partial z}$ en ce point, le terme $2\,\omega_z\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_g}{\partial z}\right)$ contribue à une circulation directe (*inverse*) au même point. Ainsi, un accroissement du vent thermique (ou ce qui revient au même, un accroissement du nombre de solénoïdes isobares-isostères) est associé à une circulation directe, une diminution, à une circulation inverse [2], le gradient vertical du vent réel étant supposé constant.

2° Lorsque le courant v_x au point O augmente (*diminue*) dans le sens x du courant, le terme $2\,\omega_x\frac{\partial v_x}{\partial x}$ contribue à une circulation transversale directe (*inverse*) lorsque le courant v_x est de NW ou de NE ($\omega_x < o$) et à une circulation inverse (*directe*) lorsque le courant v_x est de SW ou de SE ($\omega_x > o$).

On sait que l'intensité du «jet stream» des régions tempérées varie le long du courant, les maxima d'intensité se trouvant un peu en amont des thalwegs des creux bariques [5]. Par conséquent, entre une dorsale et le creux suivant (à l'E) où le courant de NW est accéléré, ainsi qu'entre un creux et la dorsale suivante (à l'E), où le courant de SW est retardé, il se produit une circulation transversale directe en relation avec la répartition de l'accélération longitudinale du courant.

3° Lorsque dans le courant v_x l'air descend, on a $\frac{d\alpha}{dt} < o$ (*monte*, $\frac{d\alpha}{dt} > o$) et le terme $-\frac{2\,\omega_x}{\alpha}\frac{d\alpha}{dt}$ contribue à une circulation directe (*inverse*) lorsque le courant v_x est de NW ou de NE et à une circulation transversale inverse (*directe*) lorsque le courant v_x est de SW ou de SE.

D'après R. G. Fleagle [1], l'air a une tendance à l'affaissement dans un «jet» de NW, tandis qu'il a une tendance à l'ascension dans un «jet» de SW. Cette circonstance contribue à une circulation transversale directe dans les deux cas.

4° Lorsque le courant v_x a une composante zonale d'W $\left(u > o, -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right.$ et $\left.\omega_y > o\right)$, le terme $2\,\omega_y\frac{\partial v_x}{\partial y}$ contribue à une circulation directe à gauche où $\left(\frac{\partial v_y}{\partial y} < o\right.$, tourbillon de glissement cyclonique dans le plan horizontal xy) et à une circulation inverse à droite (où

$\frac{\partial v_x}{\partial y} > o$, tourbillon de glissement anticyclonique dans le plan horizontal xy) du courant v_x.

Rappelons ici que, d'après C. G. Rossby [5], la circulation transversale est inverse au S d'un «jet stream» d'W, où $u > o$ et $\frac{\partial v_x}{\partial y} > o$.

Lorsque le courant v_x a une composante zonale d'E $\left(u < o,\ \frac{\pi}{2} < \beta < \frac{3\pi}{2} \text{ et } \omega_y < o\right)$, le terme $2\omega_y \frac{\partial v_x}{\partial y}$ contribue à une circulation transversale directe à droite $\left(\frac{\partial v_x}{\partial y} > o\right)$ et à une circulation inverse à gauche $\left(\frac{\partial v_x}{\partial y} < o\right)$ du courant v_x.

Il en résulte qu'au S d'un «jet stream» d'E, il y a aussi une tendance au développement d'une circulation inverse.

5° Comme le produit $u \cos \beta$ est toujours positif, quelle que soit l'orientation du courant v_x, le dernier terme, dû à un effet de latitude, contribue toujours à une circulation directe.

Nous avons décomposé l'expression de $\text{rot}_x\left(\frac{d\vec{v}}{dt}\right)$ en cinq termes dont le premier et le quatrième sont vraisemblablement les plus importants et dont le dernier est sans doute négligeable. Dans l'état actuel de nos connaissances, il est difficile de préciser davantage l'ordre de grandeur de ces termes. Une étude synoptique de la différence entre l'ascendant vertical du vent (vertical wind shear) et le vent thermique dans le «jet stream» nous paraît indispensable si l'on veut arriver à une connaissance meilleure de la circulation transversale associée aux ondulations d'un courant zonal.

Remerciements. Cet article fut rédigé à Stockholm vers la fin d'un séjour de deux mois à l'Institut Météorologique de l'Université.

Je tiens à remercier bien vivement Monsieur le Professeur C. G. Rossby de son accueil aimable, des facilités de travail qu'il m'a procurées et surtout des discussions si nombreuses et si intéressantes que nous avons eues ensemble et dont j'ai pu retirer le plus grand profit.

BIBLIOGRAPHIE.

[1] Fleagle, R. G. 1947: The fields of temperature, pressure and three-dimensional motion in selected weather situations. *Journ. of Meteor.* **4**, 6, pp. 165—185.

[2] Nyberg, A. 1949: An aerological study of large-scale atmospheric disturbances. *Tellus.* **1**, 1, pp. 44—53.

[3] Palmén, E. 1949: On the origin and structure of high-level cyclones south of the maximum westerlies. *Tellus.* **1**, 1, pp. 22—31.
Palmén, E. and Nagler K. M. 1949: The formation and structure of a large-scale disturbance in the westerlies. *Journ. of Meteor.* **6**, 4, pp. 227—242.

[4] Raethjen, P. 1949: Zyklogenetische Probleme. *Arch. Meteor. Geoph. u. Bioklim.* Ser. A, **1**, 3—4, pp. 295—346.

[5] Staff Members Dep. of Meteor. Univ. of Chicago. 1947: On the general circulation of the atmosphere in middle latitudes. *Bull Am. Meteor. Soc.* **28**, 6, pp. 255—280.

On the Vertical Distribution of Wind and Temperature over Indo-Pakistan along the Meridian 76° E in Winter

By A. M. CHAUDHURY,[1] University of Chicago

(Manuscript received 30 September, 1949)

Abstract

A mean cross-section over Indo-Pakistan along the meridian 76° E for January and February 1946 has been computed. Three jet stream centers are obtained along this meridian: one in Siberia, one just south of the plateau, and one in the tropics. The effect of the plateau gives rise to Siberian and Himalayan jets. A numerical comparison of the computed mean profile along 76° E with that of Hess (1948) along 80° W over North America is presented.

Computed wind velocity, wind shear and absolute vorticity patterns are discussed, particularly with regard to strong anticyclonic shear and very low absolute vorticity found on the south side of the Himalayan jet. The spatial distributions of the isentropic field, the gravitational stability field, the baroclinity field and the rainfall during the winter months are discussed in regard to the sense of the meridional circulation around the Himalayan and the Equatorial jets.

Introduction

Mean vertical cross sections of the average zonal wind distribution aloft across the northern hemisphere from pole to equator were first prepared by V. Bjerknes and Collaborators (1933). Although at that time the available observations were sparse and not homogeneous in space or time, the authors' sections (fig. 1) nevertheless permit us to deduce that the westerlies even in the mean are not a broad uniform current. Their momentum tends to be concentrated in a narrow meandering river of fast moving air—frequently called »jet stream» (Rossby and Collaborators 1947). More recently Willett (1944) and Hess (1948) have offered similar profiles for North America. On their sections we find the center of the jet stream near latitude 35° N at a height of about 12 km in winter. We en counter it in summer near latitude 55° N, with diminished intensity.

Cross sections made for individual days by Riehl (1948), Palmén (1948), Palmén and Nagler (1948) and mean cross sections prepared for a certain short period by Palmén and Newton (1948) have demonstrated the same feature of concentration of the westerlies aloft. As should be expected, the intensity of such instantaneous jets exceeds that of the mean sections. They also show that frequently two maxima are present in the belt of the westerlies, a fact also elaborated on by Cressman (1949). A low latitude west-wind maximum, separated from the main center of the westerlies at higher latitude, is a prominent feature of the tropical circulation over the Atlantic and Pacific oceans (1945).

Until now three different explanations have been offered for the existence of the meandering river of fast moving air. Rossby and Colla-

[1] This research was carried on when the author was studying at the University of Chicago under the sponsorship of the Government of Pakistan.

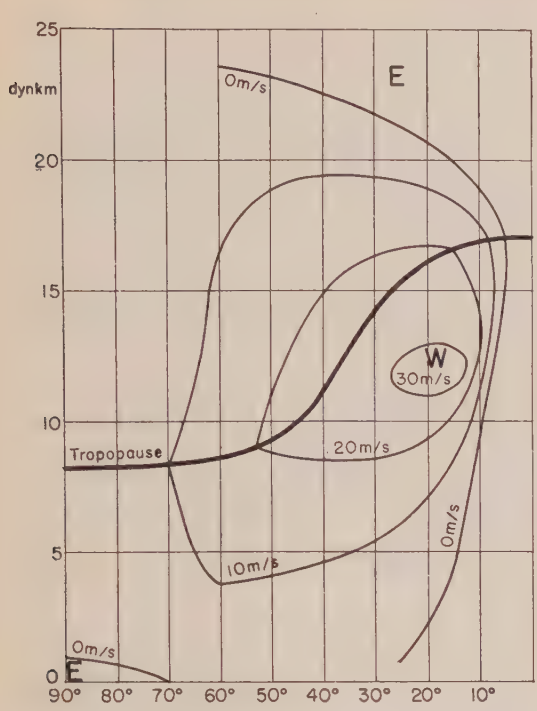


Fig. 1. Mean vertical distribution of zonal component of geostrophic wind in northern hemisphere for February, reproduced from Physikalische Hydrodynamik after Bjerknes and Collaborators (1933).

borators (1947) have advanced a theory based on the principle of conservation of absolute vorticity in connection with large scale horizontal mixing. Palmén and Newton maintain that the jet stream is associated with the polar front while Namias and Clapp (1949) have formulated the "Confluence" concept.

It is well known that the principal polar front hardly passes through Indo-Pakistan. There is, of course, evidence that disturbances enter the sub-continent from the west during the winter months. Important low-level concentrations of isotherms, however, are rarely noted in connection with such disturbances. Granted that Bjerknes' cross-section is representative, it is difficult to explain the jet stream as based on association with the polar front. In view of the foregoing discussion it is worthwhile to investigate the mean wind distribution over Indo-Pakistan in winter with recent aerological data, in order to see whether the more numerous observations now available confirm Bjerknes' section.

Available Observations

During World War II, a large number of radiosonde stations was operated over Indo-Pakistan. This network makes it possible to construct a cross section along the meridian 76° E. If Siberian data are added, the section extends from Bangalore (13° N) to Omsk (55° N).

From the observational viewpoint, January and February 1946 proved to be the most feasible period for computation. Data for India were available from the published meteorological report of the India Meteorological Department. Siberian data (500 and 700 mbs only) were obtained from the Northern Hemisphere Historical Charts of the Air Weather Service. The following stations have been used. India: (1) Bangalore (12°59′ N, 81°51′ E), (2) Poona (18°38′ N, 73°55′ E), (3) Allahabad (25°25′ N, 81°51′ E) and (4) Delhi (28°39′ N, 77°17′ E). Pakistan: (1) Peshawar (34°2′ N, 71°37′ E). U.S.S.R.: (1) Tashkent (41°20′ N, 69°18′ E), (2) Alma Ata (43°15′ N, 76°55′ E), (3) Novosibirsk (54°48′ N, 82°57′ E), (4) Omsk (54°55′ N, 73°24′ E).

Computation of the mean wind distribution was made following the standard method employed by other authors. At first, heights, temperatures, and relative humidities for standard isobaric surfaces from 1,000 mb to 200 mb were tabulated. Only those soundings that reached the 500 mb level were included. Next the arithmetic mean of all these elements was taken. From the averages of temperature and relative humidity, mean virtual temperature was computed. Thus a mean sounding for each station was obtained and heights were recalculated with the help of Bjerknes' tables (1910). These computed heights were found to be more consistent with hydrostatic equilibrium than the mean of the reported heights.

Since the flow is fairly zonal above 900 mbs, all stations were projected on their own latitude to the 76th meridian, and the wind distribution was then calculated from the geostrophic wind formula. For an isobaric surface this formula is $U = -(g/f)\left(\frac{\partial h}{\partial y}\right)_p$ where U is the zonal wind speed taken positive from the west, g the acceleration of gravity, f the Coriolis parameter, h the height of an isobaric surface, p the

pressure, and y the north-south coordinate taken positive toward north. As several of the stations are located 1000 to 2000 feet above sea level, some difficulty is encountered in the lowest levels due to extrapolation of temperature below the ground. It turned out to be the best to compute the upper wind field using the 900 mb level as reference surface with zero slope, and later add the zonal component at 1 km as obtained from the mean upper-wind charts published by the India Meteorological Department.

Field of zonal motion and absolute vorticity

The most interesting feature of the wind field (fig. 2) is the appearance of three separate areas of concentration of west wind. One center lies in Siberia, to the north of the Himalayas and the Tibetan Plateau. Its position and strength cannot be ascertained completely owing to the lack of data above 500 mbs. The second center appears immediately to the south of the plateau at about 31° N while the third lies in the tropics at 15° to 20° N. For the purpose of future reference these jets will be called Siberian, Himalayan, and Equatorial jet respectively.

In order to verify the accuracy of the computed wind profile a cross section based on the upper-wind maps of the India Meteorological Department was also prepared (fig. 3). Comparison between the two sections shows a more or less identical position of wind maxima and minima. However, the wind speeds as given by the India Meteorological Department are lower, in part presumably owing to the short pibal runs on days with strong wind or cloudiness.

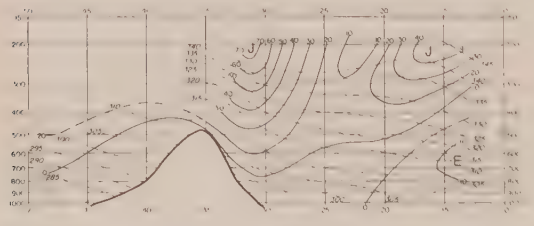


Fig. 2. Mean meridional cross-section for January and February, 1946 from Bangalore in south to Omsk in north along the meridian 76° E. Heavy solid line represents the north-south mountain profile. Thin solid lines represent constant geostrophic wind (west wind component in mps); dashed lines represent isentropes.

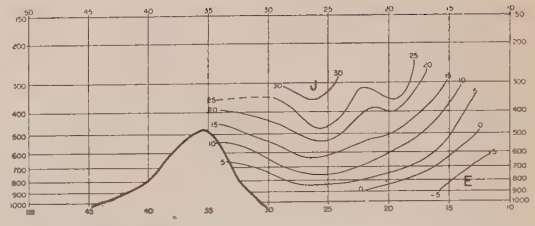


Fig. 3. Mean vertical distribution of *observed* zonal component of west wind (mps) for January along the meridian 76° E.

From the computed meridional distribution of the zonal wind the field of absolute vorticity (fig. 5) has been computed. For currents moving approximately along a latitude circle, the absolute vorticity (ζ_a) is obtained with little approximation from the relation $\zeta_a = +f - \frac{\partial U}{\partial y}$

The most interesting feature of this section is the appearance of zero values of absolute vorticity in a narrow belt 4 degrees latitude wide at the southern rim of the Himalayan jet. According to the criterion of instability for north-south acceleration of air particles moving under conservation of absolute angular momentum (Solberg 1936), little resistance can be offered by the stabilizing effect of the Coriolis force against southward acceleration in this region. Although the validity of this criterion for atmospheric motion has not yet been demonstrated generally, it will be of interest to look for evidence of descending motion near latitude 25° N, in accordance with the suggestions of Rossby and Collaborators (1947). No explanation for the region of zero absolute vorticity, however, is implied since, as also pointed out by Palmén and Newton (1948), it is not possible to obtain zero absolute vorticity in finite air columns from the operation of divergence alone.

Temperature and potential temperature

In accordance with the computed wind field, a pronounced concentration of the meridional temperature gradient (fig. 4) just south of the Himalayas is evident. Compared to other regions of the world, temperatures encountered here in the upper troposphere are remarkably low. South of the jet stream region there is an actual reversal of the meridional temperature

gradient above 400 mbs between latitudes 25° and 18° which, in a mean picture, cannot be explained easily on the basis of advective processes.

The configuration of the isentropes (fig. 2) conforms closely to the patterns emphasized by Rossby and Collaborators (1947). North of the jet center we observe a pronounced upward bulge of the isentropes and vertical crowding in the upper troposphere. On the south side, the vertical gradient is small through a deep layer of the middle troposphere. This stability distribution can be established in a more detailed way (fig. 6) by the computation of stability (S) from the relation $S = \Theta^{-1} \frac{\partial \Theta}{\partial z}$ where Θ is the potential temperature and z is the vertical coordinate taken positive upward.

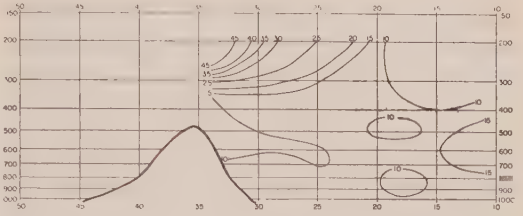


Fig. 6. Mean distribution of vertical stability (units 10^{-8} cm^{-1}). Note the general decrease of stability downward on the southside of the Himalayan jet.

Baroclinity field

In order to study the distribution of solenoids in the mean cross section the average baroclinity (fig. 7) has been computed. Baroclinity is defined as the number of solenoids per unit area (N) and is given by $-\frac{g}{T}\left(\frac{\partial T}{\partial y}\right)_p$. Maximum baroclinity appears between 400 and 300 mbs below and somewhat north of the Himalayan jet, but below and somewhat south of the Equatorial jet. Two barotropic lines are observed. One separates Himalayan and Equatorial jets with negative baroclinity above 300 mbs in correspondence with the decrease of the west wind with height between latitudes 20° and 25°. The other barotropic line lies mostly in the transition zone between easterlies and westerlies.

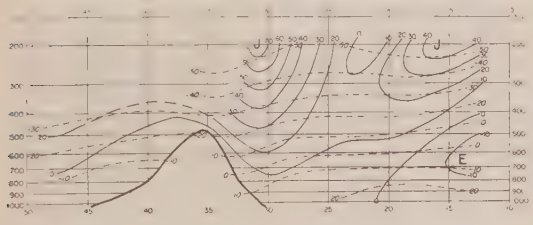


Fig. 4. Mean temperature and zonal component of geostrophic wind for January and February, 1946. Thin solid lines are isolines of westerly component of wind (mps); dashed lines represent isotherms in 0°.

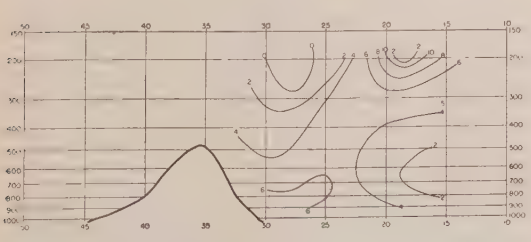


Fig. 5. Mean distribution of absolute vorticity (units 10^{-5} sec^{-1}). Note very low absolute vorticity in a latitudinal belt of 4 degrees in the high troposphere on the southside of the Himalayan jet.

Discussion of the Himalayan jet

The new computations bear out the zonal wind profile given by Bjerknes in its most important aspect—the existence of a jet stream center in the high troposphere south of the Himalayas. Beyond this point, however, there are marked differences, presumably resulting from the wide variations in longitude of the

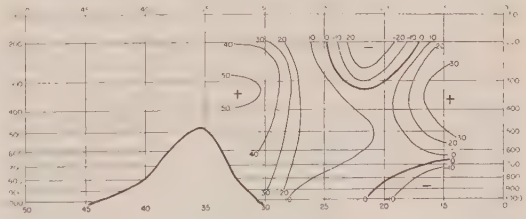


Fig. 7. Mean distribution of baroclinity (units 10^{-8} sec^{-2}). Note that the maximum baroclinity is below and somewhat north of the Himalayan jet while it is below and south of the Equatorial jet. Heavy solid lines represent zero baroclinity or barotropic lines.

stations used by Bjerknes. This observation suggests at once that there is no latitudinal symmetry around the northern hemisphere. A glance at the North American winter section of Hess (1948) confirms that this is so, provided that the unequal periods used for the computation of both cross sections are comparable. In order to bring out this asymmetry more clearly, the Hess section for 80° W was subtracted from that for 76° E (fig. 8).

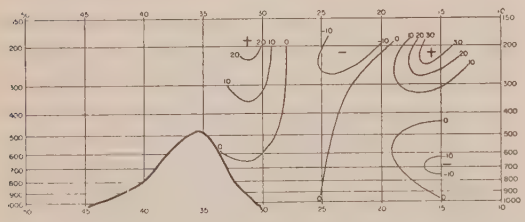


Fig. 8. Numerical comparison of zonal component of the geostropic wind along 76° E with that along 80° W after Hess (1948), the Hess section being subtracted from that along 76° E. The anomaly has been expressed in mps.

In the upper troposphere, the flow is markedly faster near latitude 30° at 76° E compared to 80° W. Because of more southerly position and greater intensity, it is easy to explain the difference in the maximum speed of the jet stream center. If the southward displacement of this center at 76° can be ascribed to the effect of the Himalayas, its position changes little from year to year, and from day to day in comparison with regions where a permanent obstacle of large size is absent. In consequence, higher wind speeds should be expected in a mean section at 76° E than at 80° W where the position of the jet is subject to much larger aperiodic variations.

It is of interest also to compute the mass transport in both sections. Total flow of mass across 80° W from 1000 to 200 mbs between latitude 10 and 65° N is $570 \cdot 10^6$ tons sec^{-1}. The latitude belt 15°—35° contributes $210 \cdot 10^6$ tons sec^{-1} (37 %) to this total. At longitude 76° E the mass transport is about the same. Although it would be necessary to study the conditions at many meridians to obtain a conclusive picture, this fact nevertheless suggests that mass transport through the troposphere is constant within latitudinal belts of limited width. At first glance fig. 8 indicates that because of the higher speeds aloft the transport should be greater; it turns out, however, that there is a compensating effect in the layer 1000—500 mbs. *Over India this compensation takes the form of the winter monsoonal current.*

It is not possible at this time to pursue this reasoning further, although it points to some interesting speculative hypotheses concerning the dynamical hypothesis of the winter monsoon. It appears permissible merely to state that the basic zonal current upon striking the plateau splits into two parts that circle the mountain ranges. It is suggested that the appearance of two west wind maxima is due to the effect of the mountain barriers. Some qualification, however, is necessary. It is well known that on many occasions a marked split of the westerly current takes place already in Europe, one branch moving along the northern rim of the continent and the other travelling across the Mediterranean Sea. Although it would be difficult to explain the observed *mean* conditions over Central Asia from this *intermittent* feature, it should certainly not be denied that on many days there is a jet stream center near latitude 30° N extending across the entire width of Eurasia.

Schematically the effect of the Himalayan plateau is represented in fig. 9. Here the solid line stands for an isobaric surface associated with a fast moving current in the initial stage before striking the plateau. The dashed

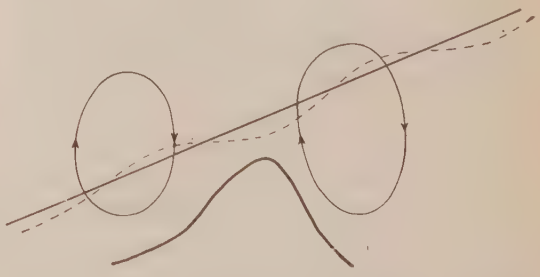


Fig. 9. Schematic representation of vertical circulation patterns around the plateau. Solid line represents an undisturbed isobaric surface associated with a mean jet stream center at about latitude 35° N; dashed line represents the deformed isobaric surface upon striking the plateau. Vertical circulation patterns are indicated by the arrows.

line represents the deformation of this isobaric surface after the current has passed through a portion of the area subject to the mountain influence. As demonstrated in the theoretical study, by the author (1949), the velocity distribution in the currents circling the mountain ranges will be such as to produce maxima in the immediate vicinity of the plateau, gradually diminishing on both sides. While this arrangement of current speed can be obtained in two-dimensional flow up to the mountain crest, this approach does not tell us why the jet along the southern border continues to intensify and sharpen well above the mountains. It is necessary to assume that a system of vertical circulations exists in the upper troposphere. Since hydrostatic equilibrium is always maintained to a high degree of approximation, this vertical circulation[1] must be in the sense indicated in fig. 9.

All supplementary cross sections support this contention. The isotherm pattern has shown us the concentration of the meridional temperature gradient just south of the mountains and an existence of an area of actual reversal of this gradient south of the jet. It is especially this latter fact that points to pronounced subsidence between latitudes 20° and 25°, since an advective hypothesis is not maintained by the mean section.

The sections of potential temperature and vertical stability bring out the effect that the field of horizontal convergence and divergence necessary to maintain the vertical circulation cell must have on the stability of air column under its influence. It is here assumed that this field of divergence has been operative for some time before the air reaches the meridian 76° E, a reasonable contention since the plateau of Pamir—»the roof of the world»—lies to the east of this meridian. North of the jet center ascent of air can account for the very low potential temperature of the high troposphere. As the ascent decreases with approach to the level of highest wind speed and the tropopause, divergence and vertical shrinking must occur from continuity requirements. Such shrinking is indicated by high values of stability in the high troposphere. After the air is displaced southward across the jet stream center, descent and stability decreasing downward should be encountered. As noted before, this stability gradient is clearly observed. This description accords with that given by Rossby and Collaborators (1947).

The observed gradient of absolute vorticity shows that the vertical motion pattern as outlined is at least dynamically possible, even though not proved, in view of the assumptions on which the Solberg instability criterion is based. However the baroclinity section offers strong support in favor of indirect circulation. If the jet stream were maintained through direct circulation, we should encounter maximum baroclinity south rather than north of the jet stream.

As final evidence we may turn to rainfall conditions during winter. If the Himalayan jet is orographically produced, its position should suffer little change from year to year as stated before. In consequence a zone of minimum rainfall should occupy central India during winter. This is born out by actual conditions (fig. 10). Noteworthy also is the narrow rainy zone extending along the northern edge of the sub-continent where "winter" depressions move eastward underneath the jet stream.

It is not possible at this time to propose a concrete explanation of the vertical circulation system. It may, however, be suggested that the surface frictional stresses of the enormous plateau area can be an important factor. This suggestion is backed by the fact that the mountains are extremely rough, and that they rise steeply along their western border from low elevations to the vicinity of the 500 mb surface or higher, so that the upper-air current must strike them at a high speed.

Discussion of the Equatorial jet

The structure of atmosphere over southern India is very different from that over the northern part of the country. Here we find a region of west wind concentration over a lower layer of easterlies. Inspection of temperature, potential temperature and baroclinity fields (figs. 2, 6 and 7) indicates that there are enough solenoids to maintain both lower easterlies and upper westerlies, provided that there is a vertical circulation system in the

[1] Such a vertical circulation may be partly due to the radiative cooling over the top of the plateau.

work-producing sense. Although the existence of such a circulation has been assumed by writers on the general circulation for many years, it was not until quite recently that the actual strength of inflow into the equatorial zone and the distribution of convergence in that region has been computed (1949). It follows from these computations, together with the continuity principle, that the previously assumed direct solenoidal circulation in low latitudes should be a reality.

This argument also finds support in the baroclinity field. In pronounced contrast to conditions over northern India maximum solenoidal concentration in the equatorial zone appears below and south of the equatorial jet. Direct vertical circulation should build up a region of west wind concentration north of the maximum as is observed.

In consequence, ascending motion should occur south of the jet stream center and descent to its north. Winter rains should be confined largely to the region south of the jet, as is indeed the case (fig. 10). It therefore appears permissible to conclude that the equatorial jet over Indo-Pakistan can be considered as an example of the "HADLEY" type (1735) of circulation.

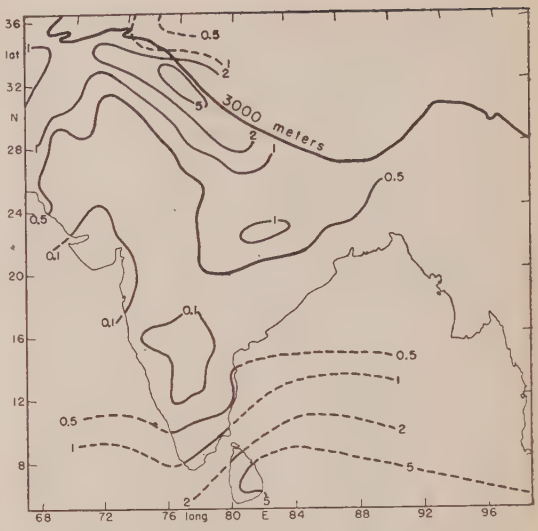


Fig. 10. Mean distribution of rainfall over Indo-Pakistan in January in inches.

Acknowledgment

The author wishes to express his gratitude to Dr. H. RIEHL for his guidance, suggestions and encouragement. Without his assistance it would have been impossible to complete this work at this time.

REFERENCES

BJERKNES V., 1910: Dynamic meteorology and hydrography. Tables 1M and 2M. Carnegie Institution of Washington, Washington.

BJERKNES, V., and COLLABORATORS, 1933: Physikalische Hydrodynamik. Berlin, J. Springer, 643—653.

CHAUDHURY, A. M., 1949: A theoretical analysis of upper-air circulation over Indo-Pakistan. To be published in *Quart. Jour. Met. Soc.*

CRESSMAN, G. P., 1949: Variations in the structure of the upper westerlies. To be published in *J. Meteor.*

HADLEY, G., 1735: Concerning the cause of the general trade winds. (London).

HESS, S. L., 1948: Some new meridional cross sections through the atmosphere. *J. Meteor.*, **5**, 293—300.

INDIA MET. DEPT., 1943: Climatological Atlas for airmen. 1—100.

NAMIAS, J., and E. CLAPP, 1949: Confluence theory of high tropospheric jet stream. To be published in *J. Meteor.*

PALMÉN, E., 1948: On the distribution of temperature and wind in the upper westerlies. *J. Meteor.*, **5**, 20—27.

— and K. NAGLER, 1948: An analysis of wind and temperature over North America in the case of approximately westerly flow. *J. Meteor.*, **5**, 58—64.

— and C. W. NEWTON, 1948: A study of the mean wind and temperature distribution in the vicinity of the polar front in winter. *J. Meteor.*, **5**, 220—226.

RIEHL, H, 1948: Jet stream in the upper troposphere and cyclone formation. *Trans. Amer. geophys. union* **29**, 175—186.

— 1945: Subtropical flow patterns in summer. *Dept. of Met., University of Chicago, Miscellaneous reports*, Met., University of Chicago, Miscellaneous reports, No. 22, 1—64.

— 1949: Unpublished studies.

ROSSBY and COLLABORATORS, University of Chicago, 1947: On the general circulation of the atmosphere in the middle latitudes. *Bull Amer. Met. Soc.*, **28**, 255—280.

SOLBERG, H., 1936: Le mouvement d'inertie de l'atmosphère stable et son rôle dans la theorie des cyclones. *Proc.-Verbaux de l'assoc. de météor., U. G. G. I., Edinburg*, 66—82.

WILLETT, H. C., 1944: Descriptive Meteorology. New York, Academic Press, 131—135.

SHORTER CONTRIBUTION

Atmospheric Effects on Cosmic Radiation, Air-mass Absorption and Meson Decay

By F. LINDHOLM, Swedish Meteorological and Hydrological Institute

(Manuscript received 17 Oktober, 1949)

Abstract

The percentage variations of twenty-four hour means of C. R. in Stockholm (magn. lat 58°) amount to ± 5 % of the mean value over a quiet period of two months' duration. The two atmospheric effects, the mass absorption due to ionisation and the meson decay, contribute ± 4 %. The residual variations, in excess of statistical fluctuations of 0.1 %, depend upon other factors, such as geomagnetic activity and solar influences.

The true absorption coefficient for zenith radiation (β) is calculated to have a value of $1.27 \cdot 10^{-3}$ cm²/g and the rate of decay (β') is found to be about 0.060 pr km. The meson producing layer is attributed to the 200 mb surface, the highest level reached by the daily radiosonde ascents in Stockholm.

The most reliable decay value is obtained by considerations of the thermal structure of air-mass changes in travelling deep depressions.

Introduction

I am indebted to Dr Malmfors for copies of the continued counter records from Stockholm (MALMFORS 1949) which have been used in the investigations discussed in this paper. The first objective, of an analysis of the intensity values Z, N and S of C. R., was to deduce the true absorption and decay coefficients of particles from the three central directions with the mean zenith distances $\vartheta = 0°$, 35° N and 35° S.

The change dN in the number of particles penetrating from a direction deviating $\vartheta°$ from the zenith depends upon ionisation and decay and can be expressed by

$$dH = -N\beta dB \sec\vartheta - N\beta' dH \sec\vartheta$$

where dB in mm Hg is the change in the air-mass over the station and dH the height-changes in m of the meson producing layer in the lower stratosphere β representing the true absorption coefficient and β' the mean rate of decay. $\beta' = \frac{1}{L}$, L the mean range of the meson before disintegration.

With the mean values $\overline{N}$, $\overline{B}$ and $\overline{H}$ for the whole period in question we have

$$N - \overline{N} = -\overline{N\beta}\,(B - \overline{B}) \sec\vartheta - \overline{N\beta'}\,(H - \overline{H'}) \sec\vartheta$$

The daily mean-values of the airmass, the barometric pressure, were taken from the hourly values of the Meteorological station at Stockholm-Bromma airport.

Tre results of counter-soundings to upper levels (PFOTZER 1936) show the meson producing layer to be at a pressure of about 100 mb. Unfortunately the daily aerological and wind soundings at Stockholm generally do not reach higher than the 200 millibar surface. These soundings, which have been employed in the present investigation, are made twice daily at 5^h and 15^h GMT. The geopotentials of the 200 millibar surface are taken from these soundings and the corresponding height values have been given in geometric metres.

The analysis of the daily means has given the following regression coefficients μ and μ' in the formula

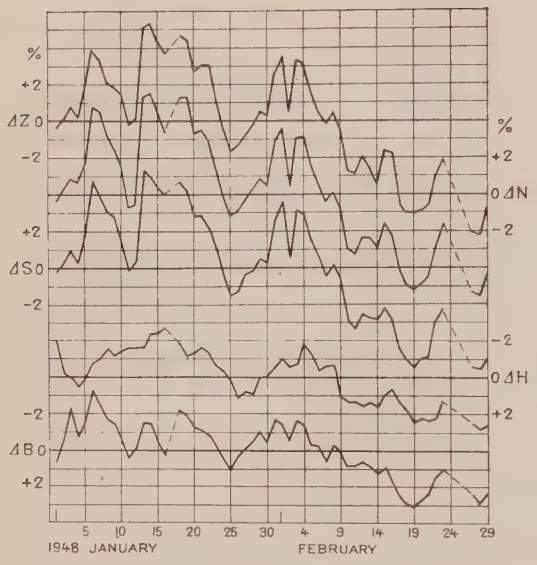


Fig. 1. Observed percentage variations of the daily means of Z, N och S intensities, the barometrique pressure at the ground and the height of the 200 mb surface.

$$N - \overline{N} = -\mu\,(B - \overline{B}) - \mu'\,(H - \overline{H}'):$$

$$Z - 25\,423 = -44.05(B - 757.3) - 1.629(H - 11\,340)$$

$$N - 16\,263 = -29.94(B - 757.3) - 0.920(H - 11\,340)$$

$$S - 15\,645 = -28.45(B - 757.3) - 0.998(H - 11\,340)$$

As shown in fig. 1 the observed variations of the daily means of Z, N and S amount to about $\pm$ 5 % of the means for the whole period. However they are reduced to $\pm$ 1 % by considering both mass absorption and decay (the thick curves in fig. 2). The thin curves in fig. 2 represent the percentage deviation after reduction to a constant pressure of 757.3 mm Hg. Other factors are responsible for residual changes larger than the statistical fluctuations of about 0.1 % of the mean intensity. Though not yet fully elucidated the influence of magnetic activity and of solar influences on the primary radiation seem to be of particular importance.

1. *Barometric coefficient.* We obtain the true absorption coefficient β from the barometric regression coefficient μ by means of the equation

$$\mu = \overline{N}\beta \sec \vartheta$$

With the above values of μ the absorption coefficients $\beta_{0°}$ and $\beta_{35°}$ for zenithal and for an incidence of 35° are

$\beta_{0°}$ = 1.73 % per cm Hg or 1.27×10^{-3} cm²/g.
$\beta_{35°}$ = 1.50 % » » » » 1.10×10^{-3} cm²/g

This result is in accordance with our knowledge of the C. R., for with a greater angle of incidence the weaker component of the C. R. is absorbed by the greater air-mass in the oblique direction.

As at sea level the hard meson-component is about 75 % of the total zenithal radiation and as the N or S intensities are found to be in about the same relation to the Z intensity, we may suppose that the oblique radiation N or S is about equivalent with the hard meson-component of the zenithal radiation. The absorption coefficient of this meson-component must then be of the order of 1.10×10^{-3} cm²/g.

The calculated values of β are lower than the *BE*-value LINDHOLM (1944) deduced without accounting for the disintegrating process, as H is also a function of B according to the hypsometrical formula.

2. *Decay coefficient.* From the regression coefficient μ' we deduce the rate of decay

for Z $\beta'_{0°}$ = 6.5 % per km and
» $N+S$ $\beta'_{35°}$ = 4.9 % » km.

Considerations of the thermal structure of travelling cyclones, causing different slopes of the isobaric surfaces at upper and lower levels, suggest a method for determination of the rate of decay independently of the barometric effect. During the period investigated there were two deep depressions passing

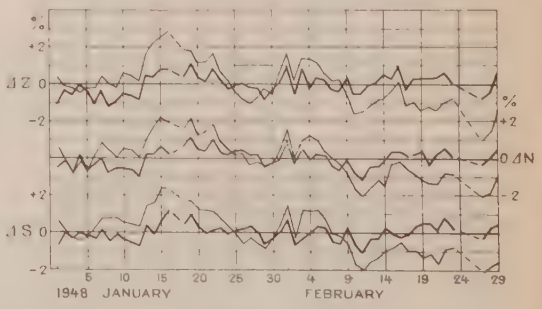


Fig. 2. Percentage variations of daily means of Z, N and S. Thin curves represent variations after reduction to the mean pressure (757.3 mm Hg). Thick curves represent variations after reduction both to mean pressure and mean height of the 200 mb. surface.

over Scandinavia, one during the days 5—7 January 1948 with a pressure minimum at Stockholm of 730.6 mm Hg at 16^h on the 6th and another one during the days 12—14 January with a minimum of 735.5 mm Hg between 0—3^h on the 14th.

Fig. 3 shows the average pressure variation at the ground and furthermore the variation of the height of the 200 mb surface and of the Z and $N + S$ values during the passages of the two depressions. The pressure and height values are taken at three hour intervals before and after the occurance of the pressure minimum at the ground. The values of Z and $N + S$ are the sum of the number of coincidenses at the same time interval on the two occasions. The points indicate the values of Z and $N + S$ reduced to a mean pressure of 737,5 mm Hg.

The corresponding observed mean values of H (200 mb) in meters at a mean pressure of 737.5 mm Hg before and after the passage of the depression and the sum of Z and $N + S$ at the two events are given in Table 1.

Table 1. Mean values of Z, N + S and height of the 200 mb surface for constant pressure at the ground during depression passages.

Hour from barometer minimum passage	Mean pressures at the station	Height of 200 mb surface meter	Z	$\mu'_{0°}$	$N + S$	$\mu'_{35°}$
12½ before	737.5	11,247	53,000		66,460	
12½ after	737.5	11,124	53,400	— 3.25	66,820	— 2.93

From the value of μ' the rates of decay are calculated

β'_z = 0.0061 % pr meter or 6.1 % pr km
β'_{N+S} = 0.0036 % » » » 3.6 % » »

If a great number of cyclone passages are treated in this manner the determination of β' can be made with great accuracy without knowing the barometric effect.

In fig. 4 the values of Z and $N + S$ reduced to 737.5 are represented as functions of H and $H \cdot \sec 35°$ resp. The correlation coefficient between Z and $N + S$ reduced to constant pressure and H^{200} mb is $\mu = -0.92$. By means of the least square method we obtain:

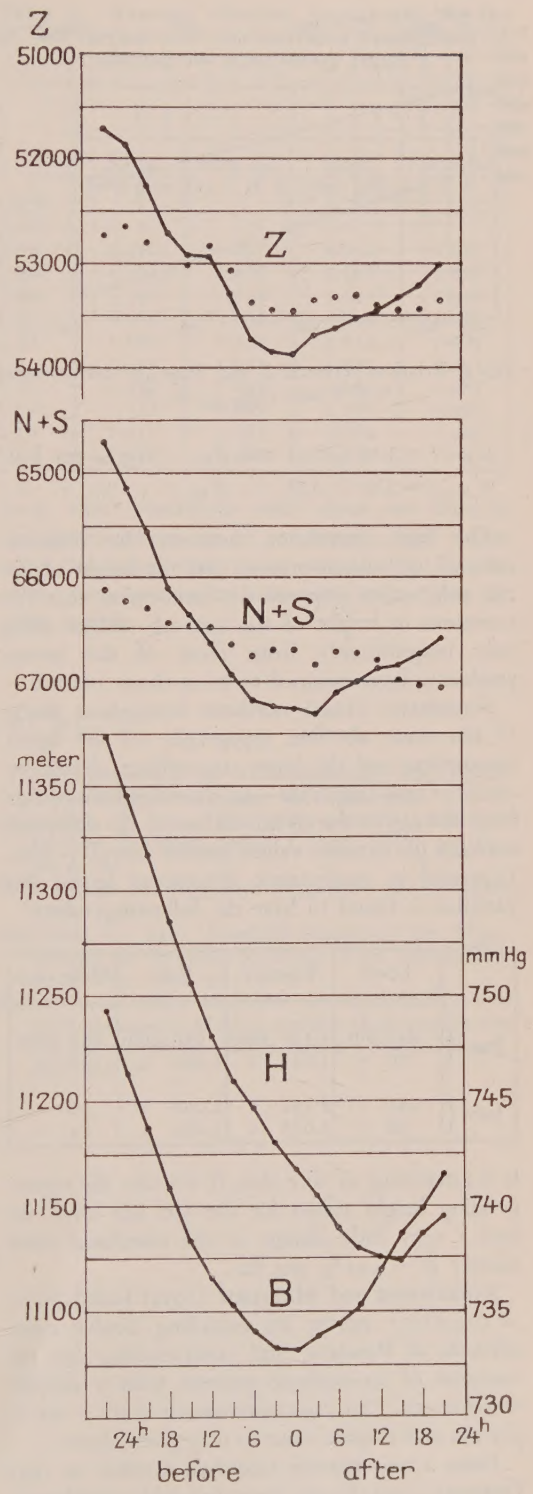


Fig. 3. Average variations of B, H, Z and $H + S$ during passages of two deep depression. (The points indicate values of Z and $N + S$ reduced to a mean pressure of 737.5 mm Hg.)

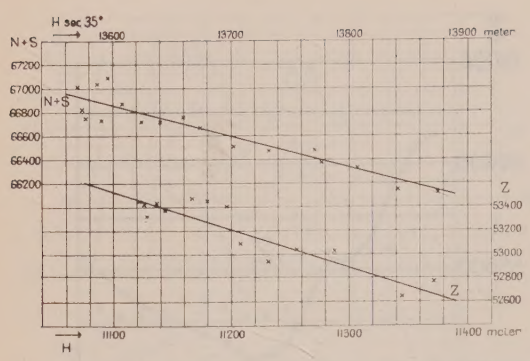


Fig. 4. Relation between Z and $H_{200\,mb}$ and between $N + S$ and $H_{200\,mb} \cdot \sec \vartheta$.

$\mu'_Z = 3.17 \pm 0.35$ and $\beta'_{0°} = 6.0\,\%$ per km
$\mu'_{N+S} = 2.60 \pm 0.28$ » $\beta'_{35°} = 3.2\,\%$ » »

The high correlation between the intensity reduced to constant pressure and the height of the 200 mb surface support the assumption that the variations in height of the 200 mb surface differ only insignificantly from those of the meson producing layer assumed to be at about 100 mb.

SCHERHAG'S (1948) northern hemisphere charts of the mean absolute topography of the upper troposphere and the lower stratosphere, drawn by means of soundings 1941—44, show but little change from the 225 to the 96 mb surface of the difference between the extreme values equator (max) — pole. Expressed in geodynamic decameters (gdm) this variation is found to have the following values:

	Level	Equator	Pole	Difference
Jan.	225 mb	1,136 gdm	980 gdm	156 gdm
	96 »	1,648 »	1,492 »	156 »
July	225 »	1,164 »	1,068 »	96 »
	96 »	1,688 »	1,624 »	64 »

It is interesting to note that, if we take the corresponding height values for the 300 mb level, we find a very little change in the calculated value namely $\beta' = 5.92\,\%$ per km.

KOLHÖRSTER and MATTHEW (1939) found $\beta'_0 = 1.13 \times 10^{-3}$ cm²/g by recording double coincidences at Potsdam, and compensating for the variation of atmospheric pressure with a variable wood screen. This value corresponds to $\beta'_0 = 6.6\,\%$ per km and is quite close to that given above.

From a two months record in London in 1943 DUPERIER (1944) found, from threefold coincidences, a lower value $\beta'_0 = 5.4\,\%$ per km at a mean pressure of 756.7 mm. This value was deduced by means of height values of the 100 mb surface over southern England. The barometric coefficent, however, was found to be 2.28 % pr cm Hg so that the sum of the two coefficients is the same in both cases.

It is interesting to find how well the height of the 200 mb surface can be determined by records of the cosmic radiation. With the decay formula given above the height values have been calculated from the intensity values of Z reduced for pressure variations. The mean error $\varepsilon\,(H) = \pm \sqrt{\frac{\Delta^2}{n-1}}$ of the individual height values is found to be $\pm$ 32 m i.e. 3 pro mille or not far from the accuracy of the actual soundings. This result was obtained by averages from the two passing cyclones.

For the whole material we find $\varepsilon\,(H) = \pm 96$ m or 8 pro mille.

3. *Temperature coefficient.* The temperature coefficient is obtained from

$$\alpha = \frac{dz}{dH} \cdot \frac{dH}{dT_m} = -\beta' \frac{dH}{dT_m}$$

If we define the mean temperatur T_m of the air column between $\overline{B}$ and the 200 mb surface by the hypsometric formula

$$H_{200\,mb} = 0.0674 \cdot T_m \log \frac{B_0}{200} \text{ km}$$

we find, for $B_0 = 1010$ mb, $\left(\frac{dH}{dT_m}\right) = 0.0474$ and with $\beta' = 0.06$ per km we have $\alpha = -0.06 \times \times 0.0474 = -0.0028$ pr °C or $\alpha = -0.28\,\%$ pr °C. We obtain for the air column between $\overline{B}$ and 100 mb $\left(\frac{dH}{dT_m}\right) = 0.00677$ and assuming the same value $\beta' = 0.06$ pr km

$$\alpha = -0.00406 \text{ or } -0.41\,\% \text{ per °C.}$$

DUPERIER (1944) has found the value of $\alpha = -0.35\,\%$ per °C. Other methods for calculation of the temperature effect have given values of α between -0.30 and -0.40 pr °C.

4. *Meson-decay and latitude effect of cosmic radiation.* The mean slope equator — pole of the isobaric surfaces in the upper troposphere and lower stratosphere, and also of the layer where mesons are

generated, must have some bearing on the interpretation of the latitude effect of C. R., which is found by CLAY (1932), HOERLIN (1933) and in the extensive observations of A. H. COMPTON (1933). From the fact that a latitude effect — an increase of the intensity of 12 per cent of the radiation —, is found, when going from the equatorial region to 55 °N geomag. latitude, it is concluded by these authors that this part of the radiation consists of corpuscles deviated by the earth's magnetic field before entering the earth's atmosphere.

In table 2 the meridional changes of the topography of the 96 mb surface are given in dyn decameter (gdm) as taken from SCHERHAGS mean chart for January and July. The great seasonal variation of the slope of the 96 mb surface with maximum in the wintermonths and minimum in summer is evident from this table.

The decay causes a variation with latitude of the mean intensity of C. R. as is shown in column 3 and 5 of table 2 for January and July respectively.

It is thus evident that in the winter months the decay effect contributes with about one third in the latitude effect, up to 55° latitude. Still farther north the meson decay must cause further increase of the average intensity amounting to 10 % in the polar region.

Table 2. Average absolute topography of the 96 mb surface over the northern hemisphere according to SCHERHAG (1948)

Lat.	January		July	
	Mean Geop. gdm.	Mean Rel. Int. C.R. Z	Mean Geop. gdm.	Mean Rel. Int. C.R. Z
90° N	1,496	1.097	1,632	1.017
80	1,504	1.088	1,636	1.015
70	1,540	1.058	1,640	1.012
60	1,564	1.049	1,644	1.010
50	1,580	1.040	1,652	1.005
40	1,596	1.033	1,670	0.994
30	1,610	1.027	1,682	0.987
20	1,624	1.013	1,676	0.990
10	1,640	1.005	1,664	0.998
0	1,648	1.00	1,660	1.00

REFERENCES

CLAY, J., BERLAGE, H. P., 1932: Variation der Ultrastrahlung mit der geographischem Breite und dem Erdmagnetismus. *Naturwiss.* **20**, 687.

COMPTON, A. A., 1933: A geographical study of Cosmic Rays. *Phys. Rev.* **43**, 387.

DUPERIER, A., 1944: A new Cosmic-Ray Recorder and the Air-Absorption and Decay of Particles. *Terr. Magn. and Atmosph. Electr.* **49**, 1.

HESS, V. F. and BENEDETTO, F. A., 1938: Mesotron Variation with Upper Air Temperatures. *Phys. Rev.* **60**, 610.

HOERLIN, H., 1933: Latitude Effect of Cosmic Radiation. *Naturwiss.* **32**, 61.

KOLHÖRSTER, W., and MATTHEWS, I., 1939: Nachweis der Mesotronen der Höhenstrahlung durch Luftdruckkompensation. *Phys. Z. S.* **40**, 142.

LINDHOLM, F., 1944: Der Barometereffekt der kosmischen Strahlung. *Arkiv f. Mat., Astr. och Fys.* **30 A**, 13.

MALMFORS, K. G., 1949: Recordings of Cosmic Radiation. *Tellus* **1**, 2.

PFOTZGER, G., 1936: Dreifachkoinzidensen der Ultrastralung aus vertikaler Richtung in der Stratosphäre. *Z. S. Phys.* **102**, 38.

SCHERHAG, R., 1948: Methoden der Wetteranalyse und Wetterprognose. Mittlere absolute Topographie der 225 mb und 96 mb Flächen über die Nordhalbkugel. Abb. 26, 27.

BOOK REVIEW

B. Gutenberg and C. F. Richter, Seismicity of the Earth and Associated Phenomena. Princeton Univ. Press, Princeton, New Jersey, 1949, 273 pp.

When two of the world's most distinguished seismologists have been working on a subject for more than ten years, one may expect important results to come out. These expectations are more than satisfied by this book, which contains the important results obtained at the Seismological Laboratory at Pasadena. The book is epoch-making in geophysics, particularly because it treats a statistically homogeneous material. This is facilitated by a consequent use of Richter's magnitude scale for earthquakes, originally introduced in 1935, and considerably developed in subsequent years by Gutenberg and Richter. Its importance lies in the fact that the magnitude is strictly defined in instrumental quantities, thus completely objective.

The reviewer is very much impressed by the large amount of work which lies behind and not less by the accuracy and conscientiousness with which it has been carried out, e. g. all the determinations taken from the International Seismological Summary have been carefully examined, as well as all other previous determinations. All seismologists in the world must be very grateful to the authors that their determinations have been compiled in the tables in the book, pp. 117—252, containing date, time, location, depth, magnitude etc. for about 3000 earthquakes, a list of the greatest value for seismological investigations. Pp. 253—267 contain a table of about 450 active volcanoes. Another feature of great help in subsequent investigations is the large list of references (pp. 104—116), containing about 560 numbers, including not only papers on seismological studies of various earthquakes (both instrumental and directly in the earthquake regions), but also on vulcanology, gravity, geology, topography etc.

After a few introducing pages we find an interesting chapter on frequency and energy of earthquakes (pp. 16—25). It gives statistical results, which thanks to the magnitude scale are more reliable than hitherto. The basic relations are one for the energy E corresponding to the magnitude M and one for the annual number N of earthquakes of magnitude M:

$$^{10}\log E = 12 + 1.8\,M$$

$$^{10}\log N = a + b\,(8 - M)$$

a, b are constants, dependent on the depth of the focus. We learn that the total number of true earthquakes may well be of the order of a million each year; that the mean annual energy released in earthquakes is about 10^{27} ergs, and that the few large earthquakes contribute most to the energy. Table 7 on p. 22 gives statistics concerning the geographical distribution of earthquakes. Among other things we find that of the shallow earthquakes (depth $\leq$ 60 km) about 80 % belong to the Circum-Pacific Belt, of intermediate earthquakes (depth 70—300 km) 91.3 % belong to the Circum-Pacific Belt and the rest to the Transasiatic Belt, and that all deep earthquakes (depth $>$ 300 km) are located to the Circum-Pacific Belt.

The greatest part of the text (pp. 28—94) is devoted to the regional discussion, where region by region round the earth is treated with regard to earthquakes, volcanoes, gravity anomalies, geology etc. The Pacific arc structures have been studied with particular care. A vertical cross-section of a typical Pacific arc is given in Fig. 6, p. 29. The book contains almost exclusively facts and very few hypotheses. But in spite of the abundance of facts it is written in an enjoyable form. The accompanying maps are excellent; one could possibly wish that they would contain more of the geographical names given in the text. The facts and the results are too many to be mentioned here. One chapter (pp. 94—97) treats tsunamis (seismic sea waves) and contains many references. Pp. 97—101 are devoted to »Mechanism», and contain several interesting ideas. A question of interest is naturally to what extent recorded earthquakes for somewhat more than 40 years are representative, in other words it would be very interesting to compare the earthquake distribution given here with that of another 40—50 years.

I hope to have given by this review the impression that the book is of the greatest value for all who have something with the earth to do in their science, geographers, geologists, geophysicists of all kinds, particularly seismologists. It is hoped that the book will be a standard work in its field for many years, but it is also hoped that it will be continued as time goes on and more data are accumulated.

Markus Båth

University of Uppsala.